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On Para-Sasakian Manifold Satisfying Certain Q Tensor with Canonical Paracontact Connection

Swati Jain | T. Raghuwanshi | M. K. Pandey | A. Goyal

Department of Mathematics, University Institute of Technology, Rajiv Gandhi Proudyogiki Vishwavidyalaya

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ABSTRACT

*The object of the present paper is to study the **Q** tensor of the canonical paracontact connection in a para-Sasakian manifold. Also, locally **Qϕ**-symmetric para-Sasakian manifold and **Qϕ**-recurrent para-Sasakian manifold with respect to the canonical paracontact connection have been studied.*

*KEYWORDS:Para-Sasakian manifold, Canonical paracontact connection, **Q** tensor.*

1. INTRODUCTION

The notion of the almost paracontact structure on a differentiable manifold defined by I. Sato [20, 21]. The structure is an analogue of the almost contact structure [6, 19] and is closely related to almost product structure (in contrast to almost contact structure, which is related to almost complex structure). Every differentiable manifold with almost paracontact structure defined by I. Sato has a compatible Riemannian metric.

An almost paracontact structure on a pseudo-Riemannian manifold M of dimension $(2n+1)$ defined by S. Kaneyuki and M. Konzai [11] and they constructed the almost paracomplex structure on $M^{2n+1} \times \mathbb{R}$. Recently, S. Zamkovoy [27] has associated the almost paracontact structure given in [11] to a pseudo-Riemannian metric of signature $(n+1, n)$ and showed that any almost paracontact structure admits such a pseudo-Riemannian metric.

As a generalization of the well-known connection defined by N. Tanaka [23] and independently by S. M. Webster [25], in context of CR-geometry, Tanaka-Webster connection was introduced by S. Tanno [24]. In a paracontact metric manifold S. Zamkovoy [27] defined a canonical connection which plays the same role of the (generalized) Tanaka-Webster connection [24] in paracontact geometry [2, 3, 4]. In this article, we study a canonical paracontact connection on a para-Sasakian manifold which seems to be the paracontact analogue of the (generalized) Tanaka-Webster connection.

STRUCTURE OF PAPER

In this paper, we consider the canonical paracontact connection on a para-Sasakian manifold and study some properties of Q tensor. This paper is organised as follows: we present a brief account of para-Sasakian manifold in section 2. The subsequent section 3 is devoted to the brief description of the canonical paracontact connection and its properties. In section 4, we study the Q tensor of canonical paracontact connection in a para-Sasakian manifold. In section 5, a locally $Q\phi$-symmetric para-Sasakian manifold with respect to the canonical paracontact connection. In section 6, is studied that $Q\phi$-recurrent para-Sasakian manifold with respect to the canonical paracontact connection.

2. PRELIMINARIES

Let M be a differentiable manifold of dimension $2n+1$. If there exists a triple (ϕ,ξ,η) of a tensor field ϕ of type (1,1), a vector field ξ and a 1-form η on M^{2n+1} which satisfies the relations [11]:

$$\phi^2 = I - \eta \otimes \xi, \tag{2.1}$$

$$\eta(\xi) = 1,\ \phi\xi = 0, \tag{2.2}$$

$$\eta \circ \phi = 0,\ rank(\phi) = 2n, \tag{2.3}$$

where I denotes the identity transformation, then we say the triple (ϕ,ξ,η) is an almost paracontact structure and the manifold is an almost paracontact manifold.

Moreover, the tensor field ϕ induces an almost paracomplex structure on the paracontact distribution $D = ker\eta$, i.e, the eigen distributions $D^{\pm}$ corresponding to the eigenvalues ± 1 of ϕ are both n-dimensional.

If an almost paracontact manifold M with an almost paracontact structure (ϕ,ξ,η) admits a pseudo-Riemannian metric g such that [27]

$$g(\phi X, \phi Y) = -g(X,Y) + \eta(X)\eta(Y),\ X,Y \in \Gamma(TM), \tag{2.4}$$

then we say that M is an almost paracontact metric manifold with an almost paracontact metric structure (ϕ,ξ,η,g) and such metric g is called compatible metric. Any compatible metric g is necessarily of signature $(n+1,n)$.

From (2.4), one can see that [27]

$$g(X,\phi Y) = -g(\phi X, Y), \tag{2.5}$$

$$g(X,\xi) = \eta(X), \tag{2.6}$$

for any $X,Y \in \Gamma(TM)$.

The fundamental 2-form of M is defined by

$$\phi(X,Y) = g(X,\phi Y).$$

An almost paracontact metric structure becomes a paracontact metric structure [27]

if

$$g(X,\phi Y) = d\eta(X,Y),$$

for all vector field X,Y,where

$$d\eta(X,Y) = \frac{1}{2}[X\eta(Y) - Y\eta(X) - \eta([X,Y])].$$

For a (2n+1) dimensional manifold M with the structure (ϕ,ξ,η,g), one can also construct a local orthonormal basis which is called a ϕ-basis $(X_i,\phi X_i,\xi),(i=1,2,\dots,n)$ [27].

An almost paracontact metric structure (ϕ,ξ,η,g) on M is a para-sasakian manifold if and only if [27]

$$(\nabla_X \phi)Y = -g(X,Y)\xi + \eta(Y)X, \tag{2.7}$$

for any $X,Y \in \Gamma(TM)$ and ∇ is Levi-Civita connection of M.

From (2.7), it can be seen that

$$\nabla_X \xi = -\phi X. \tag{2.8}$$

A P-Sasakian manifold satisfying

$(\nabla_X \eta)(Y) = -g(X,Y) + \eta(X)\eta(Y)$,(2.9)

is called an special para-Sasakian manifold or briefly a SP-Sasakian manifold [1].

Example 2.1 *[5]. Let $M = \mathbb{R}^{2n+1}$ be the $(2n+1)$-dimensional real number space with $(x_1,y_1,x_2,y_2,\dots\dots x_n,y_n,z)$ standard coordinate system. Defining*

$$\phi\frac{\partial}{\partial x_\alpha} = \frac{\partial}{\partial y_\alpha},\quad \phi\frac{\partial}{\partial y_\alpha} = \frac{\partial}{\partial x_\alpha},\quad \phi\frac{\partial}{\partial z} = 0,$$

$$\xi = \frac{\partial}{\partial z}, \eta = dz,$$

$$g = \eta \otimes \eta + \sum_{\alpha=1}^{n} dx_\alpha \otimes dx_\alpha - \sum_{\alpha=1}^{n} dy_\alpha \otimes dy_\alpha,$$

where $\alpha = 1,2,\dots,n$, then the set (M,ϕ,ξ,η,g) is an almost paracontact metric manifold.

In a para-Sasakian manifold M, the following relations hold [27] :

$$g(R(X,Y,Z),\xi) = \eta(R(X,Y,Z)) = g(X,Z)\eta(Y) - g(Y,Z)\eta(X), \tag{2.10}$$

$$R(X,Y,\xi) = \eta(X)Y - \eta(Y)X, \tag{2.11}$$

$$R(\xi,X,Y) = -g(X,Y)\xi + \eta(Y)X, \tag{2.12}$$

$$R(\xi,X,\xi) = X - \eta(X)\xi, \tag{2.13}$$

$$S(X,\xi) = -2n\eta(X), \tag{2.14}$$

$$S(\phi X,\phi Y) = S(X,Y) + (n-1)\eta(X)\eta(Y), \tag{2.15}$$

for any vector fields $X,Y,Z \in \Gamma(TM)$. Here, R is Riemannian curvature tensor and S is Ricci tensor defined by $g(QX,Y) = S(X,Y)$, where Q is the Ricci operator.

Quasi Einstein manifolds, introduced by M. C. Chaki and R. K. Maity [7], are natural generalizations of Einstein manifolds. According to them, a non-flat Riemannian manifold $(M,g)(n>2)$ is a quasi-Einstein manifold [7] if its Ricci tensor S of type (0,2) is not identically zero and satisfies the following condition

$$S(X,Y) = ag(X,Y) + bA(X)A(Y), \tag{2.16}$$

for all vector fields X and Y, where a and b are scalars with $b \neq 0$. A is a non-zero 1-form such that

$$g(X,\xi) = A(X), \tag{2.17}$$

for all vector fields X and ξ being a unit vector. Quasi Einstein manifoldshave been studied by several authors such as U. C. De and G. C. Ghosh [8], U. C. De and B. K. De [9] and U. C. De et. al. [10] and many others.

3.PARA-SASAKIAN MANIFOLDS WITH CANONICAL PARACONTACT CONNECTION

In this section, we give a brief account of paracontact connetion and study it on a para-Sasakian manifold.

Now, we consider the connection $\tilde{\nabla}$ defined by [24],

$$\tilde{\nabla}_X Y = \nabla_X Y + \eta(X)\phi Y - \eta(Y)\nabla_X \xi + (\nabla_X \eta)Y \cdot \xi, \tag{3.1}$$

where $X, Y \in \Gamma(TM)$ and ∇ denotes Levi-Civita connetion on M.

In view of (2.8) in (3.1), we arrive at

$$\tilde{\nabla}_X Y = \nabla_X Y + \eta(X)\phi Y + \eta(Y)\phi X + g(X,\phi Y)\xi. \tag{3.2}$$

Definition 3.1 On a para-Sasakian manifold, the connetion $\tilde{\nabla}$ given by (3.2) is called a canonical paracontact connetion.

On a para-Sasakian manifold, canonical paracontact connection $\tilde{\nabla}$ has the following properties:

$$\tilde{\nabla}\eta = 0, \tilde{\nabla}g = 0, \tilde{\nabla}\xi = 0, \tag{3.3}$$

$$(\tilde{\nabla}_X \phi)Y = (\nabla_X \phi)Y + g(X,Y)\xi - \eta(Y)X. \tag{3.4}$$

The curvature tensor $\tilde{R}$ of a para-Sasakian manifold M with respect to the canonical paracontact connection $\tilde{\nabla}$ is defined by

$$\tilde{R}(X,Y,Z) = \tilde{\nabla}_X \tilde{\nabla}_Y Z - \tilde{\nabla}_Y \tilde{\nabla}_X Z - \tilde{\nabla}_{[X,Y]} Z. \tag{3.5}$$

If we use equation (3.2) in (3.5), we get

$$\begin{aligned}\tilde{R}(X,Y,Z) &= R(X,Y,Z) + g(Y,Z)\eta(X)\xi - g(X,Z)\eta(Y)\xi \\ &+\eta(Y)\eta(Z)X - \eta(X)\eta(Z)Y + 2g(X,\phi Y)\phi Z \\ &+g(X,\phi Z)\phi Y - g(Y,\phi Z)\phi X,\end{aligned}$$

where R and $\tilde{R}$ are the curvature tensors of M with respect to the LeviCivita connetion ∇ and canonical paracontact connetion $\tilde{\nabla}$ respectively.

Assume that T and $\tilde{T}$ are curvature tensors of type (0,4) defined by

$$T(X,Y,Z,W) = g(R(X,Y,Z),W),$$

and

$$\tilde{T}(X,Y,Z,W) = g(\tilde{R}(X,Y,Z),W),$$

respectively.

Theorem 3.1 In a para-Sasakian manifold the following relations hold:

$$\tilde{R}(X,Y,Z) + \tilde{R}(Y,Z,X) + \tilde{R}(Z,X,Y) = 0, \tag{3.7}$$

$$\tilde{T}(X,Y,Z,W) + \tilde{T}(Y,X,Z,W) = 0, \tag{3.8}$$

$$\tilde{T}(X,Y,Z,W) + \tilde{T}(X,Y,W,Z) = 0, \tag{3.9}$$

$$\tilde{T}(X,Y,Z,W) - \tilde{T}(Z,W,X,Y) = 0. \tag{3.10}$$

Suppose that $E_i = \{e_i, \phi e_i, \xi\}(i = 1,2,\dots,n)$ is a local orthonormal ϕ-basis of a para-Sasakian manifold M. Then the Ricci tensor $\tilde{S}$ and the scalar curvature $\tilde{r}$ of M with respect to canonical paracontact connection $\tilde{\nabla}$ are defined by

$$\tilde{S}(X,Y) = \sum_{i=1}^{n} g\big(\tilde{R}(e_i,X,Y),e_i\big) - \sum_{i=1}^{n} g\big(\tilde{R}(\phi e_i,X,Y),\phi e_i\big) +g(\tilde{R}(\xi,X,Y),\xi) \tag{3.11}$$

and

$$\tilde{r} = \sum_{j=1}^{n} \tilde{S}(e_j,e_j) - \sum_{j=1}^{n} \tilde{S}(\phi e_j,\phi e_j) + \tilde{S}(\xi,\xi), \tag{3.12}$$

respectively.

Theorem 3.2 In a para-Sasakian manifold M, the Ricci tensor $\tilde{S}$ and scalar curvature $\tilde{r}$ of canonical paracontact connection $\tilde{\nabla}$ are defined by

$$\tilde{S}(X,Y) = S(X,Y) - 2g(X,Y) + (2n+2)\eta(X)\eta(Y), \tag{3.13}$$

$$\tilde{r} = r - 2n, \tag{3.14}$$

where S and r denote the Ricci tensor and scalar curvature of Levi-Civita connection ∇, respectively. Consequently, $\tilde{S}$ is symmetric.

Lemma 1 If M is a para-Sasakian manifold with canonical paracontact connection $\tilde{\nabla}$, then

$$g(\tilde{R}(X,Y,Z),\xi) = \eta(\tilde{R}(X,Y,Z)) = 0, \tag{3.15}$$

$$\tilde{R}(X,Y,\xi) = \tilde{R}(\xi,X,Y) = \tilde{R}(\xi,X,\xi) = 0, \tag{3.16}$$

$$\tilde{S}(X,\xi) = 0, \tag{3.17}$$

for all $X, Y, Z \in \Gamma(TM)$.

4. Q TENSOR

In this section, we consider the Q tensor of para-Sasakian manifold in canonical paracontact connection.

In 2012, Mantica and Molinari [13, 14] defined a generalized (0,2) symmetric Z tensor expressed as

$$Z(X,Y) = S(X,Y) + \psi g(X,Y), \tag{4.1}$$

where S denotes the Ricci tensor and ψ is an arbitrary scalar function. It is referred to the generalized Z tensor simply as the Z tensor.

In 2013 , Mantica and Suh [15] introduced a new type of tensor whose trace is the Z tensor. The Q tensor is defined as

$$Q(X,Y,V) = R(X,Y,V) - \frac{\psi}{(n-1)}[g(Y,V)X - g(X,V)Y], \quad (4.2)$$

where R denotes the curvature tensor and ψ is an arbitrary scalar function.

The $(0,4)Q$ tensor is expressed as

$$Q(X,Y,V,U) = R(X,Y,V,U) - \frac{\psi}{(n-1)}[g(Y,V)g(X,U) - g(X,V)g(Y,U)]. \quad (4.3)$$

If we put ξ for X, Y and V respectively in the equation (4.2), then in view of the equations (2.6), (2.11) and (2.12), we get

$$Q(\xi,Y,V) = \left[1 + \frac{\psi}{(n-1)}\right][\eta(V)Y - g(Y,V)\xi], \quad (4.4)$$

$$Q(X,\xi,V) = \left[1 + \frac{\psi}{(n-1)}\right][g(X,V)\xi - \eta(V)X], \quad (4.5)$$

and

$$Q(X,Y,\xi) = \left[1 + \frac{\psi}{(n-1)}\right][\eta(X)Y - \eta(Y)X]. \quad (4.6)$$

Similar to the definition (4.2), we define the $\tilde{Q}$ tensor of paracontact connection $\tilde{\nabla}$ in para-Sasakian manifold by

$$\tilde{Q}(X,Y,V) = \tilde{R}(X,Y,V) - \frac{\psi}{(n-1)}[g(Y,V)X - g(X,V)Y], \quad (4.7)$$

Also, the type $(0,4)$ tensor field $\tilde{Q}$ is given by

$$\tilde{Q}(X,Y,V,U) = \tilde{R}(X,Y,V,U) - \frac{\psi}{(n-1)}[g(Y,V)g(X,U) - g(X,V)g(Y,U)], \quad (4.8)$$

for the arbitrary vector fields X, Y, V, U

With the help of equation (3.6) in (4.7), we get

$$\tilde{Q}(X,Y,V) = R(X,Y,V) + g(Y,V)\eta(X)\xi - g(X,V)\eta(Y)\xi + \eta(Y)\eta(V)X - \eta(X)\eta(V)Y + 2g(X,\phi Y)\phi V + g(X,\phi V)\phi Y - g(Y,\phi V)\phi X - \frac{\psi}{(n-1)}[g(Y,V)X - g(X,V)Y].$$

which using the equation (4.2) in the above equation, yields

$$\tilde{Q}(X,Y,V) = Q(X,Y,V) + [g(Y,V)\eta(X) - g(X,V)\eta(Y)]\xi + [\eta(Y)X - \eta(X)Y]\eta(V) + 2g(X,\phi Y)\phi V + g(X,\phi V)\phi Y - g(Y,\phi V)\phi X - \frac{\psi}{(n-1)}[g(Y,V)X - g(X,V)Y \quad (4.9)$$

Now, taking ξ for each of the vector field X, Y and V in the above equation and using equations (2.2), (2.3), (2.6), (4.4), (4.5) and (4.6), we get

$$\tilde{Q}(\xi,Y,V) = \frac{\psi}{(n-1)}[\eta(V)Y - g(Y,V)\xi], \quad (4.10)$$

$$\tilde{Q}(X,\xi,V) = \frac{\psi}{(n-1)}[g(X,V)\xi - \eta(V)X], \quad (4.11)$$

and

$$\tilde{Q}(X,Y,\xi) = \frac{\psi}{(n-1)}[\eta(X)Y - \eta(Y)X], \quad (4.12)$$

Taking inner product with ξ in the equation (4.2) and using (2.6), (2.10), we get

$$\eta\big(Q(X,Y,V)\big) = \left[1 + \frac{\psi}{(n-1)}\right][g(X,V)\eta(Y) - g(Y,V)\eta(X)], \quad (4.13)$$

Similarly, from the equations (2.2), (2.3), (4.9) and (4.13), we obtain

$$\eta(\tilde{Q}(X,Y,V)) = \frac{\psi}{(n-1)}[g(X,V)\eta(Y) - g(Y,V)\eta(X)]. \quad (4.14)$$

5. LOCALLY $Q\phi$-SYMMETRIC PARA-SASAKIAN MANIFOLD WITH CANONICAL PARACONTACT CONNECTION

In section 5, a locally $Q\phi$ -symmetric para-Sasakian manifold with respect to the canonical paracontact connection.

Definition 5.1 An $(2n+1)$-dimensional para-Sasakian manifold M^{2n+1} is said to be locally Qφ-symmetric if

$$\phi^2((\nabla_U Q)(X,Y,V)) = 0,$$

for all vector fields X, Y, U and V.

Definition 5.2 An (2n+1)-dimensional para-Sasakian manifold M^{2n+1} is said to be locally Qφ-symmetric with respect to the canonical paracontact connection if

$$\phi^2\left((\tilde{\nabla}_U \tilde{Q})(X,Y,V)\right) = 0,$$

for all vector fields X, Y, U and V orthogonal to ξ, where $\tilde{Q}$ is the Q-tensor of the canonical paracontact connection $\tilde{\nabla}$.

Theorem 5.1 A para-Sasakian manifold is locally Qφ-symmetric with respect to the canonical paracontact connection $\tilde{\nabla}$ if and only if it is so with respect to the Levi-Civita connection ∇.

Proof. From equation (3.2), we have

$$(\tilde{\nabla}_U\tilde{Q})(X,Y,V) = (\nabla_U\tilde{Q})(X,Y,V) + \eta(U)\varphi\tilde{Q}(X,Y,V) \\ +\eta(\tilde{Q}(X,Y,V))\varphi U + g(U,\varphi\tilde{Q}(X,Y,V))\xi. \tag{5.1}$$

Now, differentiating equation (4.9) covariantly with respect to U, we get

$$\begin{aligned}(\nabla_U\tilde{Q})(X,Y,V) = (\nabla_U Q)(X,Y,V) + 2g(X,\phi Y)(\nabla_U\phi)(V) \\ -\eta(X)(\nabla_U\eta)(V)Y - \eta(V)(\nabla_U\eta)(X)Y \\ +g(X,\phi V)(\nabla_U\phi)(Y) - g(Y,\phi V)(\nabla_U\phi)(X) \\ +[g(Y,V)(\nabla_U\eta)(X) - g(X,V)(\nabla_U\eta)(Y)]\xi \\ +[\eta(Y)(\nabla_U\eta)(V)X + \eta(V)(\nabla_U\eta)(Y)X],\end{aligned} \tag{5.2}$$

In the above equation using (2.1), (2.2), (2.6), (2.9), (2.10), (4.2), (4.9), (4.14) , we get

$$\begin{aligned}&(\tilde{\nabla}_U\tilde{Q})(X,Y,V)) = (\nabla_U Q)(X,Y,V) - g(Y,V)g(X,U)\xi + \\ &g(X,V)g(Y,U)\xi + g(Y,V)\eta(X)\eta(U)\xi - \\ &g(X,V)\eta(Y)\eta(U)\xi - \eta(Y)g(U,V)X + 2\eta(Y)\eta(U)\eta(V)X - \\ &\eta(V)g(Y,U)X + \eta(X)g(V,U)Y - 2\eta(X)\eta(U)\eta(V)Y + \\ &\eta(V)g(X,U)Y + 2g(X,\phi Y)\eta(V)U - 2g(X,\phi Y)g(U,V)\xi + \\ &g(Y,\phi V)g(U,X)\xi - g(Y,\phi V)\eta(X)U + g(X,V)\eta(U)\phi Y + \\ &g(X,\phi V)\eta(Y)U - g(X,\phi V)g(U,Y)\xi + g(Y,V)\eta(U)\phi X + \\ &\eta(Y)\eta(U)\eta(V)\phi X - \eta(X)\eta(U)\eta(V)\phi Y + 2g(X,\phi Y)\eta(U) + \\ &g(X,\phi V)\eta(U)Y - g(X,\phi V)\eta(U)\eta(Y)\xi - g(Y,\phi V)\eta(U)X + \\ &g(Y,\phi V)\eta(U)\eta(X)\xi + g(X,V)g(U,\phi Y)\xi - \\ &g(Y,V)g(U,\phi X)\xi + g(U,\phi X)\eta(Y)\eta(V)\xi - \\ &g(U,\phi Y)\eta(X)\eta(V)\xi + 2g(X,\phi Y)g(U,V)\xi - \\ &2g(X,\phi Y)\eta(U)\eta(V)\xi + 2g(X,\phi V)g(U,Y)\xi - \\ &2g(X,\phi V)\eta(Y)\eta(U)\xi - 2g(X,\phi Y)\eta(U)\eta(V)\xi - \\ &g(Y,\phi V)g(U,X)\xi + g(Y,\phi V)\eta(X)\eta(U)\xi - \\ &\tfrac{\psi}{(n-1)}\{g(Y,V)\phi X + g(X,V)\phi Y\}\eta(U) + \tfrac{\psi}{(n-1)}\{g(X,V)\eta(Y) - \\ &g(Y,V)\eta(X)\}\phi U - \tfrac{\psi}{(n-1)}\{g(Y,V)g(U,\phi X) + \\ &g(X,V)g(U,\phi Y)\}\xi,\end{aligned} \tag{5.3}$$

Applying ϕ^2 on both sides of the above equation and using equations (2.1), (2.2), we get

$$\begin{aligned}&\phi^2((\tilde{\nabla}_U\tilde{Q})(X,Y,V) = \phi^2((\nabla_U Q)(X,Y,V) - \eta(Y)g(U,V)X + \\ &\eta(X)\eta(Y)g(U,V)\xi + 2\eta(U)\eta(Y)\eta(V)X - \\ &2\eta(U)\eta(Y)\eta(V)\eta(X)\xi - \eta(V)g(U,Y)X + \\ &\eta(V)g(U,Y)\eta(X)\xi + \eta(X)g(U,V)Y - 2\eta(X)\eta(U)\eta(V)Y + \\ &2\eta(X)\eta(U)\eta(V)\eta(Y)\xi + \eta(V)g(U,X)Y - \\ &\eta(V)g(U,X)\eta(Y)\xi + 2g(X,\phi Y)\eta(V)U - \\ &2g(X,\phi Y)\eta(V)\eta(U)\xi + g(X,\phi V)\eta(Y)U - \\ &g(X,\phi V)\eta(Y)\eta(U)\xi - g(Y,\phi V)\eta(X)U + \\ &g(Y,\phi V)\eta(X)\eta(U)\xi + g(X,V)\eta(U)\phi Y + g(Y,V)\eta(U)\phi X + \\ &\eta(U)\eta(Y)\eta(V)\phi X - \eta(U)\eta(X)\eta(V)\phi Y + \\ &2g(X,\phi Y)\eta(U)V - 2g(X,\phi Y)\eta(U)\eta(V)\xi + \\ &g(X,\phi V)\eta(U)Y - g(X,\phi V)\eta(U)\eta(Y)\xi - g(Y,\phi V)\eta(U)X + \\ &g(Y,\phi V)\eta(U)\eta(X)\xi - \eta(X)g(U,V)\eta(Y)\xi + \\ &\tfrac{\psi}{(n-1)}\{g(X,V)\phi Y - g(Y,V)\phi X\}\eta(U) + \tfrac{\psi}{(n-1)}\{g(X,V)\eta(Y) - \\ &g(Y,V)\eta(X)\}\phi U,\end{aligned}$$

Now if X, Y, U, V are orthogonal to ξ, then the above equation reduces to

$$\phi^2((\tilde{\nabla}_U\tilde{Q})(X,Y,V)) = \phi^2((\nabla_U Q)(X,Y,V)),$$

This completes the proof.

6. $Q\phi$-RECURRENT PARA-SASAKIAN MANIFOLD WITH CANONICAL PARACONTACT CONNECTION

In section 6, is studied that $Q\phi$ -recurrent para-Sasakian manifold with respect to the canonical paracontact connection.

Definition 6.1 An (2n+1)-dimensional para-Sasakian manifold M^{2n+1} is said to be Qφ-recurrent if

$$\varphi^2((\nabla_U Q)(X,Y,V)) = A(U)Q(X,Y,V),$$

for the arbitrary vector fields X, Y, U and V, where A is a non zero 1-form.

Definition 6.2 An (2n+1)-dimensional para-Sasakian manifold M^{2n+1} is said to be Qφ-recurrent with respect to the canonical paracontact connection if

$$\varphi^2(\tilde{\nabla}_U\tilde{Q})(X,Y,V) = A(U)\tilde{Q}(X,Y,V), \tag{6.1}$$

for arbitrary vector fields X, Y, U and V.

Theorem 6.1 A Qφ-recurrent para-Sasakian manifold with respect to a canonical paracontact connection is a quasi Einstein manifold.

Proof. From equations (2.1) and (6.1) , we have

$$(\tilde{\nabla}_U\tilde{Q})(X,Y,V) - \eta((\tilde{\nabla}_U\tilde{Q})(X,Y,V))\xi = A(U)\tilde{Q}(X,Y,V),$$

which reduces to

$$g\left((\tilde{\nabla}_U\tilde{Q})(X,Y,V),W\right) - \eta\left((\tilde{\nabla}_U\tilde{Q})(X,Y,V)\right)\eta(W) \\ = A(U)g(\tilde{Q}(X,Y,V),W).$$

Using equations (2.2), (2.3), (2.6), (4.9) and (5.3) in the above equation, we get

$$g((\nabla_U Q)(X,Y,V),W) - \eta(Y)g(U,V)g(X,W) + 2g(X,W)\eta(V)\eta(U)\eta(Y) - g(X,W)g(U,Y)\eta(V) + g(U,V)g(Y,W)\eta(X) - 2g(Y,W)\eta(V)\eta(U)\eta(X) + g(Y,W)g(U,X)\eta(V) + 2g(X,\phi Y)g(U,W)\eta(V) + g(X,\phi V)g(U,W)\eta(Y) - g(Y,\phi V)g(U,W)\eta(X) + g(X,V)g(\phi Y,W)\eta(U) + g(Y,V)g(\phi X,W)\eta(U) + g(\phi X,W)\eta(U)\eta(Y)\eta(V) - g(\phi Y,W)\eta(U)\eta(X)\eta(V) + 2g(X,\phi Y)g(V,W)\eta(U) - 4g(X,\phi Y)\eta(U)\eta(W)\eta(V) + g(X,\phi V)g(Y,W)\eta(U) - g(Y,\phi V)g(X,W)\eta(U) - 3g(X,\phi V)\eta(U)\eta(W)\eta(Y) + 2g(Y,\phi V)\eta(U)\eta(W)\eta(X) - \eta(\nabla_U Q)(X,Y,V)\eta(W) + g(Y,U)\eta(V)\eta(W)\eta(X) - g(X,U)\eta(V)\eta(W)\eta(Y) - \frac{\psi}{(n-1)}g(Y,V)\{g(\phi X,W)\eta(U) + g\phi U,W\eta X+\psi n-1 gX,V\{g\phi Y,W\eta U+g\phi U,W\eta Y\}=A(U)g(Q(X,Y,V),W) + A(U)g(Y,V)\eta(W)\eta(X) - A(U)g(X,V)\eta(W)\eta(Y) + A(U)g(X,W)\eta(V)\eta(Y) - A(U)g(Y,W)\eta(V)\eta(X) + 2A(U)g(X,\phi Y)g(\phi V,W) + A(U)g(X,\phi V)g(\phi Y,W) - A(U)g(Y,\phi V)g(\phi X,W),$$

Again using equations (2.9) and (4.2), we get

$$\begin{aligned}
&2g(X,W)\eta(V)\eta(U)\eta(Y) \\
&- g(X,W)\{g(U,V)\eta(Y) + g(U,Y)\eta(V)\} \\
&+ g(Y,W)g(U,V)\eta(X) - g(Y,W)\eta(U)\eta(X)\eta(V) \\
&+ g(Y,W)g(U,X)\eta(V) - g(Y,W)\eta(U)\eta(X)\eta(V) \\
&+ 2g(X,\phi Y)g(U,W)\eta(V) + 2g(X,\phi V)g(U,W)\eta(Y) \\
&- g(Y,\phi V)g(U,W)\eta(X) + g(X,V)g(\phi Y,W)\eta(U) \\
&+ g(Y,V)g(\phi X,W)\eta(U) + g(\phi X,W)\eta(U)\eta(Y)\eta(V) \\
&- g(\phi Y,W)\eta(U)\eta(X)\eta(V) + 2g(X,\phi Y)g(V,W)\eta(U) \\
&- 4g(X,\phi Y)\eta(U)\eta(W)\eta(V) + g(X,\phi V)g(Y,W)\eta(U) \\
&- g(Y,\phi V)g(X,W)\eta(U) - 3g(X,\phi V)\eta(U)\eta(Y)\eta(W) \\
&+ 2g(Y,\phi V)\eta(U)\eta(X)\eta(W) + g(U,Y)\eta(V)\eta(X)\eta(W) \\
&- g(U,X)\eta(V)\eta(Y)\eta(W) + g\big((\nabla_U R)(X,Y,V),W\big) \\
&- \eta\big((\nabla_U R)(X,Y,V)\big)\eta(W) \\
&- \frac{\psi}{(n-1)}\{g(Y,V)g(\phi X,W) - g(X,V)g(\phi Y,W)\}\eta(U) \\
&+ \frac{\psi}{(n-1)}[g(Y,V)(\nabla_U\eta)(X)\eta(W) - g(X,V)(\nabla_U\eta)(Y)\eta(W)] \\
&- \frac{\psi}{(n-1)}[g(Y,V)g(\nabla_U X,W) - g(X,V)g(\nabla_U Y,W)] \\
&= A(U)g(R(X,Y,V),W) + A(U)g(Y,V)\eta(X)\eta(W) \\
&- A(U)g(X,V)\eta(Y)\eta(W) + A(U)g(X,W)\eta(Y)\eta(V) \\
&- A(U)g(Y,W)\eta(X)\eta(V) + 2A(U)g(X,\phi Y)g(\phi V,W) \\
&+ A(U)g(X,\phi V)g(\phi Y,W) - A(U)g(Y,\phi V)g(\phi X,W) \\
&- \frac{\psi}{(n-1)}[A(U)g(Y,V)g(X,W) - A(U)g(X,V)g(Y,W)],
\end{aligned}$$

Let $\{e_i\}, i = 1,2,\dots,2n$ be an orthonormal basis of the tangent space at every point of the manifold. Then putting $X = W = e_i$ in the above equation and taking summation over $\{e_i, \phi e_i, \xi\}, 1 \le i \le 2n$, we get

$$\begin{aligned}
(\nabla_U S)(Y,V) - \eta((\nabla_U R)(e_i,Y,V))\eta(e_i) &+ 4n\eta(Y)\eta(V)\eta(U) \\
&- 2ng(U,V)\eta(Y) - 2ng(U,Y)\eta(V) \\
&+ g(U,V)\eta(Y) - 2\eta(Y)\eta(V)\eta(U) \\
&- 2ng(Y,\phi V)\eta(U) \\
&+ g(U,Y)\eta(V)g(\phi Y,V)\eta(U) \\
&+ 2g(V,\phi Y)\eta(U) - \eta(Y)\eta(V)\eta(U) \\
&+ g(Y,U)\eta(V) + 2g(Y,\phi V)\eta(U) \\
&+ \frac{\psi}{(n-1)}g(\phi U,V)\eta(Y) \\
&+ \frac{\psi}{(n-1)}[g(U,Y)\eta(V) - \eta(Y)\eta(V)\eta(U)] \\
&= A(U)S(Y,V) + g(V,Y)A(U) \\
&- A(U)\eta(Y)\eta(V) + 2nA(U)\eta(Y)\eta(V) \\
&- A(U)\eta(Y)\eta(V) \\
&- \frac{\psi}{(n-1)}[2ng(V,Y)A(U) \\
&- g(V,Y)A(U)] + 3A(U)g(\phi Y,\phi V),
\end{aligned}$$

Putting $V = \xi$ in the above equation and using equations (2.2), (2.4), (2.6), (2.14), yields

$$(\nabla_U S)(Y,\xi) - \eta((\nabla_U R)(e_i,Y,\xi))\eta(e_i) + \frac{\psi}{(n-1)}[g(U,Y) - \eta(Y)\eta(U)] + (2n-2)\eta(Y)\eta(U) - (2n-2)g(U,Y) + \frac{\psi}{(n-1)}(2n-1)\eta(Y)A(U) = 0. \tag{6.2}$$

The second term on L.H.S. of equation (6.2) takes the form

$$E = \eta\big((\nabla_U R)(e_i,Y,\xi)\big)\eta(e_i) = g\big((\nabla_U R)(e_i,Y,\xi),\xi\big)g(e_i,\xi),$$

which is denoted by λ. In this case λ vanishes. Namely, we have

$$g\big((\nabla_U R)(e_i,Y,\xi),\xi\big) = g(\nabla_U R(e_i,Y,\xi),\xi) - g(R(\nabla_U e_i,Y,\xi),\xi) - g(R(e_i,\nabla_U Y,\xi),\xi) - g(R(e_i,Y,\nabla_U\xi),\xi) \tag{6.3}$$

at p $\in M^{2n+1}$. In the local coordinates $\nabla_U e_i = U^j\Gamma_{ji}^h e_h$, where Γ_{ji}^h are the Christoffel symbols. Since $\{e_i\}$ is an orthonormal basis, the metric tensor $g_{ij} = \delta_{ij}$, where δ_{ij} is the Kronecker delta and hence the Christoffel symbols are zero. Therefore $\nabla_U e_i = 0$.

Also, we have

$$g(R(e_i, \nabla_U Y, \xi), \xi) = 0, \tag{6.4}$$

Since R is the skew symmetric. Using equation (6.4) and $\nabla_U e_i = 0$ in equation (6.3), we get

$$g((\nabla_U R)(e_i, Y, \xi), \xi) = g(\nabla_U R(e_i, Y, \xi), \xi) - g(R(e_i, Y, \nabla_U \xi), \xi).$$

In view of $g(R(e_i, Y, \xi), \xi) = -g(R(\xi, \xi, Y), e_i) = 0$ and $\nabla_U g = 0$, we have

$$g(\nabla_U R(e_i, Y, \xi), \xi) + g(R(e_i, Y, \xi), \nabla_U \xi) = 0,$$

which implies

$$g((\nabla_U R)(e_i, Y, \xi), \xi) = -g(R(e_i, Y, \xi), \nabla_U \xi) - g(R(e_i, Y, \nabla_U \xi), \xi).$$

Since R is skew symmetric, we have

$$g((\nabla_U R)(e_i, Y, \xi), \xi) = 0. \tag{6.5}$$

Using equation (6.5) in equation (6.2), we get

$$(\nabla_U S)(Y, \xi) = -\frac{\psi}{(n-1)}[g(Y, U) - \eta(Y)\eta(U) + (2n-1)\eta(Y)A(U)] - (2n-2)\eta(Y)\eta(U) + (2n-2)g(Y, U).$$

Now, we have

$$(\nabla_U S)(Y, \xi) = \nabla_U S(Y, \xi) - S(\nabla_U Y, \xi) - S(Y, \nabla_U \xi),$$

which on using equations (2.1), (2.3), (2.9) and (2.14) takes the form

$$(\nabla_U S)(Y, \xi) = 2ng(U, Y) + S(U, Y). \tag{6.7}$$

From equations (6.6) and (6.7), we have

$$S(Y, U) = -\frac{\psi}{(n-1)}[g(Y, U) - \eta(Y)\eta(U) + (2n-1)\eta(Y)A(U)] - (2n-2)\eta(Y)\eta(U) - 2g(Y, U),$$

Replacing Y and U by ϕY and ϕU respectively in the above equation and using equations (2.3), (2.4), (2.15), we get

$$S(Y, U) = \left[\frac{\psi + 2n - 2}{n-1}\right] g(U, Y) - \left[\frac{\psi + n^2 - 1}{n-1}\right] \eta(U)\eta(Y),$$

which shows that it is a quasi Einstein manifold.

Theorem 6.2 In a Qφ-recurrent para-Sasakian manifold M^{2n+1} admitting canonical paracontact connection, the characteristic vector field ξ and the vector field ϱ associated with 1 -form A are co-directional and the 1-form A is given by

$$A(U) = \eta(\rho)\eta(U).$$

Proof. By virtue of equations (2.1) and (6.1), we have

$$(\tilde{\nabla}_U \tilde{Q})(X, Y, V) = \eta((\tilde{\nabla}_U \tilde{Q})(X, Y, V))\xi + A(U)\tilde{Q}(X, Y, V).$$

Using equations (2.2), (2.3), (4.9) and (5.3) in the above equation, we get

$$\begin{aligned}
(\nabla_U Q)(X, Y, V) &- \eta(Y)g(U, V)X(\nabla_U Q)(X, Y, V) \\
&- \eta(Y)g(U, V)X(\nabla_U Q)(X, Y, V) \\
&- \eta(Y)g(U, V)X(\nabla_U Q)(X, Y, V) \\
&- \eta(Y)g(U, V)X + \eta(V)g(X, U)Y \\
&+ 2g(X, \phi Y)\eta(V)U - 2g(X, \phi Y)g(U, V)\xi \\
&+ g(X, \phi V)\eta(Y)U - g(X, \phi V)g(Y, U)\xi \\
&+ g(Y, \phi V)g(U, X)\xi - g(Y, \phi V)\eta(X)U \\
&+ g(X, V)\eta(U)\phi Y + g(Y, V)\eta(U)\phi X \\
&- \frac{\psi}{(n-1)} g(Y, U)\eta(V)\phi X \\
&+ \frac{\psi}{(n-1)} g(X, V)\eta(U)\phi Y \\
&- 2g(X, \phi Y)\eta(U)\eta(V)\xi \\
&+ 2g(X, \phi V)\eta(U)Y - g(X, \phi V)\eta(U)\eta(Y)\xi \\
&- g(Y, \phi V)\eta(U)X \\
&+ \frac{\psi}{(n-1)} g(X, V)\eta(Y)\phi U \\
&- \frac{\psi}{(n-1)} g(Y, V)\eta(X)\phi U \\
&= \eta((\nabla_U Q)(X, Y, V))\xi \\
&- \eta(Y)\eta(X)g(V, U)\xi - \eta(X)\eta(V)g(Y, U)\xi \\
&+ \eta(X)\eta(Y)g(V, U)\xi + \eta(V)\eta(Y)g(X, U)\xi \\
&+ 2g(X, \phi Y)\eta(V)\eta(U)\xi \\
&- 2g(X, \phi Y)g(V, U)\xi \\
&+ g(X, \phi V)\eta(Y)\eta(U)\xi \\
&- g(X, \phi V)g(Y, U)\xi + g(Y, \phi V)g(X, U)\xi \\
&- g(Y, \phi V)\eta(X)\eta(U)\xi \\
&+ 2g(X, \phi V)\eta(Y)\eta(U)\xi \\
&- g(X, \phi V)\eta(Y)\eta(U)\xi \\
&- g(Y, \phi V)\eta(X)\eta(U)\xi + A(U)Q(X, Y, V) \\
&+ [g(Y, V)\eta(X) - g(X, V)\eta(Y)]A(U)\xi \\
&+ [\eta(Y)X - \eta(X)Y]\eta(V)A(U) \\
&+ 2g(X, \phi Y)A(U)\phi V + g(X, \phi V)A(U)\phi Y \\
&- g(Y, \phi V)A(U)\phi X,
\end{aligned}$$

Taking the inner product of the above equation with ξ and using equation (2.2), (4.13), we get

$$A(U)\left[1+\frac{\psi}{(n-1)}\right][g(X,V)\eta(Y)-g(Y,V)\eta(X)] + A(U)[g(Y,V)\eta(X)-g(X,V)\eta(Y)] = 0.$$

Writing two more equations by the cyclic permutation of U, X and Y from the above equation and adding them to above equation, we get

$$\frac{\psi}{(n-1)}[A(U)g(X,V)\eta(Y)-A(U)g(Y,V)\eta(X)] + \frac{\psi}{(n-1)}[A(X)g(Y,V)\eta(U) - A(X)g(U,V)\eta(Y)] + \frac{\psi}{(n-1)}[A(Y)g(U,V)\eta(X) - A(Y)g(X,V)\eta(U)] = 0.$$

Putting $Y = V = e_i$ in the above equation and taking summation over i, $1 \leq i \leq 2n+1$, we get

$$A(U)\eta(X) = A(X)\eta(U),$$

for all vector fields X and U. Replacing X by ξ in the above equation, we get

$$A(U) = \eta(\rho)\eta(U),$$

for all vector fields U, where $A(\xi) = g(\xi,\rho) = \eta(\rho), \rho$ being the vector field associated to the 1 -form A, i.e.

$$g(X,\rho) = A(X),$$

This completes the proof.

Conflict of interest statement

Authors declare that they do not have any conflict of interest.

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