*As per UGC guidelines an electronic bar code is provided to seure your paper*

*International Journal for Modern Trends in Science and Technology*, 8(09): 285-289, 2022

ISSN: 2455-3778 online
DOI: https://doi.org/10.46501/IJMTST0809053
Available online at: http://www.ijmtst.com/vol8issue09.html

# A study on the links between music and mathematics with respect to interval, scales, tuning and temperament

**Kabir Gupta**

**To Cite this Article**
Kabir Gupta . A study on the links between music and mathematics with respect to interval, scales, tuning and temperament. International Journal for Modern Trends in Science and Technology 2022, 8(09), pp. 285-289. https://doi.org/10.46501/IJMTST0809053

**Article Info**
Received: 02 September 2022; Accepted: 20 September 2022; Published: 25 September 2022.

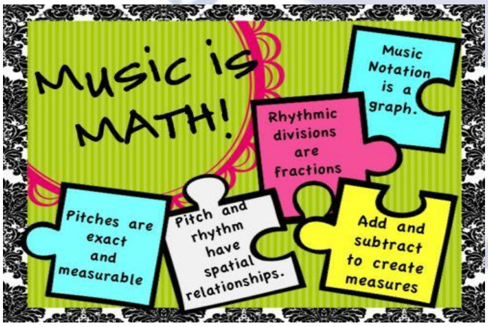
**ABSTRACT**

***This paper looks closely at the links between music and mathematics. On the face of it, both of them look poles apart. On closer analysis, there are a lot of basic mathematical functions which are used repeatedly in making music. Some of the examples are the use of the Pythagoras theorem, mathematical progression. These are some of the areas which show an extremely close relationship between the two. An attempt has been made to analyze as many links as possible between the two so to say completely different sciences.***

***Research question: The aim is to understand the close link between music and mathematics. This would be attempted keeping the following basic musical properties in mind. Some of these that have been attempted are interval, scales, temperament and tuning in perspective.***

## 1. INTRODUCTION

Figure 1: Interrelationship between music and mathematics




Source: Google image

Music, as the word suggests, seems far removed from mathematics. One thinks of music as a harmonious blend of form, expression and emotion to produce the most beautiful synchronization of sound. It is one of the most universal cultural aspects of all human societies. The best manner to describe this would be by using these quotes; "Life is one grand, sweet song so start the music", "Music is the shorthand of emotion", and "Music is the wine that fills the cup of silence". Music theory analyzes the pitch, timing and structure of music. Without the aid of mathematics, music is like a headless chicken. To study elements in music such as tempo, chord, progression, form and meter, the science of mathematics is required. As the quote by Pythagoras goes, "There is geometry in the humming of strings; there is music in the spacing of spheres".

## 2.DEFINITION

At the outset, the first reaction on a music mathematics combination is that they are so far removed from each other. But on a closer look, it is apparent that they are quite close to each other. Looking at the definitions of each, we find thatthe connect between the two is extremely close.

2.1. Music

Music can be defined as vocal and instrumental sounds which have been combined in a way to produce beauty of form, harmony and expression of emotion. It is the art of arranging sounds in time through the elements of melody, harmony and timbre. The definition of music would definitely take into account elements such as pitch, rhythm and dynamics. It is also very often defined, as the art of ordering tone and sounds whether in succession or in combination, but definitely in a continuous relationship, all to produce a composition which has unity and continuity.

2.2- Mathematics

Mathematics on the other hand is an abstract science of number, quantity and space which can be used in the abstract form (pure mathematics) or when it is applied to other disciplines, like music, then it is used to enhance the quality and understanding of the underlying discipline.

Music and mathematics are two peas in the same pod because together they create the most beautiful synchronized elements that give immense pleasure to oneself and others. Time signatures, beats per minute formulaic progressions, are all familiar words with respect to music. Performing music reinforces the above and is part of the brain which is involved in solving mathematical problems.

## 3.THE SIMILARITY BETWEEN MUSIC AND MATHEMATICS

The biggest similarity between the two is in the form of patterns. Music uses definite repetition in the form of verses and choruses in the course of singing or playing an instrument. While one is playing music, there is a chorus or a verse which is continuously repeated. In the case of mathematics, there is a pattern that one tends to follow which could be based on a mathematical formula and is often applied in solving problems. At times, the formulas used are typical ones based on the problems on hand. These could include geometry, differential calculus and maybe trigonometry. The two disciplines have a deep commonality. This really means that the relationship between the two is quite far and deep. An attempt would be made to look at this similarity with respect to

1. Interval
2. Temperament
3. Tuning
   - i) Pythagorean tuning
   - ii) Just intonation
   - iii) Mean tone
4. Scales

### 3.1. Interval

Any student being introduced to music is told that he or she has to learn by heart certain sets of diatonic intervals (the intervals that run from seconds to the sevenths). In music theory, an interval is a difference in pitch between two sounds. An interval may be described as **horizontal, linear or melodic** if it refers to successively sounding tones like two pitches which are adjacent to another in a melody, these could be **vertical or harmonic**, if it concerns simultaneously sounding tones such as chord. Diatonic scale is very commonly used in western music where the interval is the difference between the notes of a diatonic scale. The smallest of such intervals are called semitone and any interval smaller than that is known as microtone. At times, intervals can be arbitrarily small, and could possibly go unnoticed by the human ear.

An interval in physical terms is defined as the ratio between two sonic frequencies. What this essentially means is that there is a successive increment of pitch by the same interval leading to an exponential increase of frequency. The human ear perceives this as a linear increase, but in actuality, it is an exponential increaseThe type of scales that are studied in music are basically the *chromatic scale* (twelve notes), the *whole-tonescale* (six notes), the *pentatonic scale*(five notes), the *octatonic or diminished scales* (eight notes). An octave is a musical instrument defined by the ratio 2:1 regardless of the starting frequency, i.e. one could start from 100 Hz to 200 Hz or from 2000 Hz to 4000 Hz, both would be called octave. The intervals which are generally most consonant (combination of two or more tones of different frequency that result in a musically pleasing sound) to the human ear are intervals represented by small integer ratio.

### 3.2. Temperament

In discussing the temperament of a musical instrument, interval is an important concept that has to be taken into account. A temperament is a tuning system that compromises the pure intervals of just intonation (accuracy of pitch in playing or singing on a stringed instrument e.g. a guitar, it is a variation in the pitch) to meet other requirements.

Those that have small interval ratios are known as *Just intervals* (musical intervals in whole number ratios such as 3:2 or 4:3 is said to be pure) and the temperament is known as *Just temperament* (a musical interval which maintains exact integer ratios between pitches for example the ratio 3:2 is called Just musical $5^{th}$ or perfect $5^{th}$). Similarly, the manner in which Just intervals are prevalent, so also are Just intonations, which require the tuning of the musical instrument for a specific key. Humans find two notes pleasant or consonant when their pitches form a simple ratio.

Table 1- The comparison of musical notes with frequency ratio:

| Interval | Example notes | Frequency ratio |
|---|---|---|
| Octave | C-C′ | 1:2 |
| Fifth | C-G | 2:3 |
| Fourth | C-F | 3:4 |
| Major third | C-E | 4:5 |
| Minor third | A-C | 5:6 |

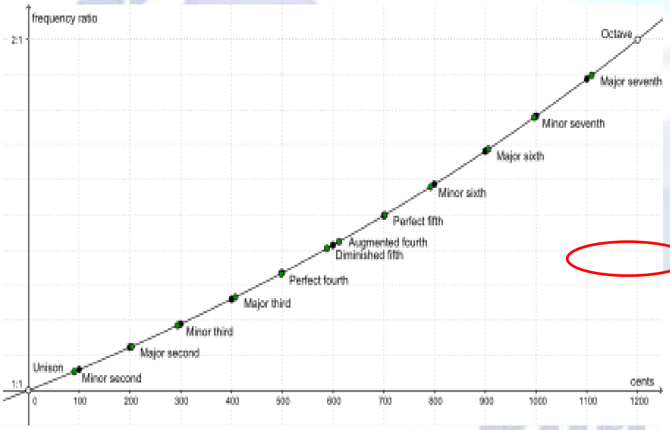
Source: Google image

Irrational relations such as 1:√ 2 (whose simplest good rational approximation is32: 45) sounds cacophonous or dissonant to our ears.

### 3.3. Tuning

3.3.1. Pythagorean tuning

Figure 2: Depiction of Pythagorean tuning




Source: Google image

The figure above indicates the relationship between frequency ratio and cent (a cent is a unit of measure for the ratio between two frequencies, the interval between two adjacent piano keys is 100 cents, an octave has a frequency ratio of 2:1 -spans twelve semitones (100 cents is half a step) which is equivalent to 1200 cents). Frequency ratio which is indicated on the Y axis is used to describe intervals based on the tuning system.

Pythagorean tuning is a system, which is the oldest tuning system and was introduced by the mathematician Pythagoras (sixth century B.C). The geometrically motivated Pythagorean tuning is based on a frequency ratio 1:2 of the octave and the ratio 2:3 of the fifth. Intervals which are derived from these ratios can be obtained by adding and subtracting fifths and octaves, and the resultant frequency ratio involves powers of two or powers of three. The easiest to tune by ear is the ratio 3:2 (also known as pure perfect fifth), which is why importance is given to the integer three. Pythagoras did not actually study the frequencies that make up pleasing intervals that is termed as music, nor did he study the relationship between frequencies and musical scale. He only made observations about the length of the string that made these intervals or scales. Pythagoras made remarkable contribution to the mathematical theory of music. He played the musical instrument Lyre (string instrument). His main contribution to music was the relationship between the ratio of the length of the string and whole numbers. This theory was extended to other musical instruments. He used his prowess over the musical instrument lyre as a means to help those who were ill. The ratio that he discovered between the strings led to a sound which was harmonious. He also discovered that the sound and harmonic frequencies could actually heal ailing patients and was the first to use music as a medicine.

Table 2: Definitions of intervals and their relations

| Δ | Interval name | Interval | JI ratio | Pyt. ratio |
|---|---|---|---|---|
| 0 | (Perfect) unison | C4 – C4 | 1:1 | 1:1 |
| 1 | Minor second | C4 – D♭4 | 15:16 | $3^5:2^8$ |
| 2 | Major second | C4 – D4 | 8:9 | $2^3:3^2$ |
| 3 | Minor third | C4 – E♭4 | 5:6 | $3^3:2^5$ |
| 4 | Major third | C4 – E4 | 4:5 | $2^6:3^4$ |
| 5 | (Perfect) fourth | C4 – F4 | 3:4 | $3:2^2$ |
| 6 | Tritone | C4 – F♯4 | 32:45 | $2^9:3^6$ or $3^6:2^{10}$ |
| 7 | (Perfect) fifth | C4 – G4 | 2:3 | 2:3 |
| 8 | Minor sixth | C4 – A♭4 | 5:8 | $3^4:2^7$ |
| 9 | Major sixth | C4 – A4 | 3:5 | $2^4:3^3$ |
| 10 | Minor seventh | C4 – B♭4 | 5:9 | $3^2:2^4$ |
| 11 | Major seventh | C4 – B4 | 8:15 | $2^7:3^5$ |
| 12 | (Perfect) octave | C4 – C5 | 1:2 | 1:2 |

Source: Muller, FMP, Springer 2015.

The table above indicates a clear difference with respect to just intonation ratio (JI ratio) and the Pythagorean ratio (Pythagorean Ratio)

3.3.2 Just Intonation

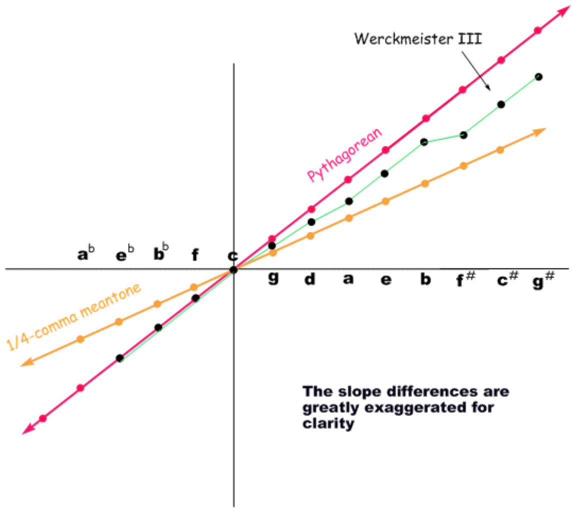
This was discovered by the composer Harry Partch, who defined his own scale with 43 pitches to the octave and invented his own instruments

This is a system of tuning in which the correct size of all the intervals of the scale is calculated by different additions and subtractions of pure natural numbers third and fifth (the intervals that occur between the fourth and fifth , second and third tones of the natural harmonic series). The harmonic series in mathematics is an infinite series formed bythe summation of positive unit fractions. The modern unfretted bowed instruments like the violin and the cello play in just intonation. They can modulate to any key center. This can also be played with brass instruments which do not have valves e.g. the bugle. Unfortunately just intonation makes it impossible to change keys or tune your guitar. Just intonation suffers from inequality of intervals.

3.3.3 Mean tone temperament

This is a musical temperament which is also a tuning system obtained by narrowing the fifths so that the ratio is less than 3:2. This results in them being narrower than a perfect fifth (as indicated in red in figure 2) and results in the third closer to the pure. Fifths are closer to the center harmonically as compared to thirds (this is a series of musical tones whose frequencies are integral multiples of the frequency. These are notes which are produced in a special way.) It is closely related to the harmonic progression in mathematics where the vibrations in music indicate simple harmonics relationship to each other. Harmonic progression (HP) is a sequence of real numbers determined by taking reciprocals of the arithmetic progression that does not contain zero.

Figure 3: Comparison between Pythagorean and ¼-comma mean tone and Werckmeister III




Source: Google Image

¼ comma (comma's are the difference in size between two semitones) mean-tones or ¼ c narrows each fifth in a series of fifths by 1/4th of a syntonic comma (a small comma type interval between two notes is equal to the frequency ratio of 81:80)

Werckmeister III

This type of tuning was described by Andreas Werckmeister in his writings. The tuning systems were numbered in two different ways, one was known as good temperament and the second depended on the labeling on his monochord (a one chord musical instrument).

Thus the mean tone temperament represents an attempt to avoid the problems that one might face during Pythagorean tuning or just intonation. There is a tendency to narrow the fifth so that the purity of the common thirds is maintained.

**3.4 Scales**

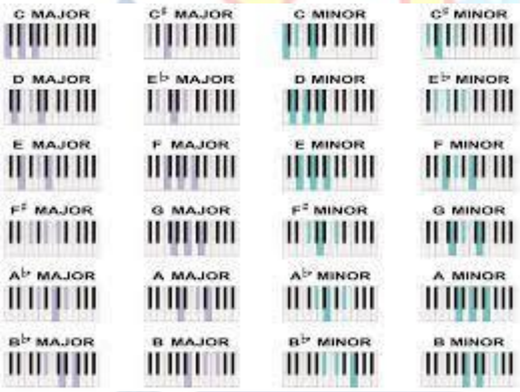
Diatonic scale is a type of music scale which has seven notes, and these are called heptatonic scale that includes five whole steps (whole tones) and two half steps (semitones) in each octave. The two half steps may be separated from each other by two or three whole steps, thus diatonic scales are a step wise arrangement forming an octave. Most scales are diatonic and they include major and minor, the harmonic minor is an exception. Amongst these, the major scale is the most familiar and easily recognizable of all diatonic scales. On a Piano keyboard starting with C and playing all the white notes, it would result in not only a major scale, but a diatonic scale.

Figure 4: Diatonic scale


Source: Google image

Major and minor scales are essentially diatonic and they form the basis of all musical instruments, and vocals. All of them indicate a mathematical pattern. This is essentially the basis of any 'music'.

Figure 5: Major and minor scales




Source: Google image

The music created in the form of diatonic scale also indicates a leaning towards the use of mathematics in the symmetric pattern that they follow.

## 4. CONCLUSION

Music theory as we know it studies the pitch, timing and structure of music. Mathematics is used extensively to study elements of music, for example- tempo, chord progression, form and meter. In trying to communicate new ways of composing and hearing music, the use of abstract algebra and number theory has come into play. The basis of musical sound exhibits an array of number properties. Operations such as transposition and inversion often called isometrics, all come under the purview of mathematics and they essentially preserve the intervals between tones in a set.

Some theorists, while using musical set theory, have used abstract algebra to analyze music. They form a part of an abelian, which can be defined as having members related by a commutative operation, in this case, a group with twelve elements. Transformational theory (a branch of mathematics) has also been used to form a branch of music theory. It was developed by David Lewin. Many music theorists have in fact proposed a number of musical applications based on sophisticated algebraic concepts. In fact, composers have used the *golden ratio* and *Fibonacci* numbers in their work.

The famous mathematician and musicologist GuerinoMazzola has in fact used category theory (Topos theory) for the basis of music, this includes prop ology as a basis for rhythm and motives, and differential geometry as a basis of musical frequency, tempo and intonation.

Thus one can clearly see the close link that music has with mathematics, and as time progresses, the use of one field in the other and vice versa is becoming more and more widespread.

## Conflict of interest statement

Authors declare that they do not have any conflict of interest.

## REFERENCES

[1] Benson, D. (2007). Music: a mathematical offering. United Kingdom: Cambridge University Press.

[2] Fry, H. (2015). The Mathematics of Love: Patterns, Proofs, and the Search for the Ultimate Equation. United States: Simon & Schuster/ TED.

[3] Harkleroad, L. (2006). The Math Behind the Music. United Kingdom: Cambridge University Press.

[4] Loy, G. (2011). Musimathics: The Mathematical Foundations of Music. United Kingdom: MIT Press.

[5] Maor, E. (2020). Music by the Numbers: From Pythagoras to Schoenberg. United States: Princeton University Press.

[6] Mazzola, G., Müller, S., Göller, S. (2012). The Topos of Music: Geometric Logic of Concepts, Theory, and Performance. Switzerland: Springer Basel AG.

[7] Roberts, G. E. (2016). From Music to Mathematics: Exploring the Connections. United States: Johns Hopkins University Press.

[8] Rothstein, E. (1996). Emblems of the Mind. United States: HarperCollins.

[9] Wright, D. (2009). Mathematics and Music. United States: American Mathematical Society.