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International Journal for Modern Trends in Science and Technology, 8(03): 85-88, 2022

ISSN: 2455-3778 online
DOI: https://doi.org/10.46501/IJMTST0803015
Available online at: http://www.ijmtst.com/vol8issue03.html

Pendant Domination in Line Graphs

Dr. Anil R. Sedamkar

Department of Science, Government Polytechnic Afzalpur, Dist. Kalaburagi, Karnataka, India
Corresponding author e-mail: anil.sedamkar@gmail.com

To Cite this Article
Dr. Anil R. Sedamkar. Pendant Domination in Line Graphs. International Journal for Modern Trends in Science and Technology 2022, 8(03), pp. 85-88. https://doi.org/10.46501/IJMTST0803015

Article Info
Received: 06 February 2022; Accepted: 08 March 2022; Published: 14 March 2022.

ABSTRACT

A set $D \subseteq V(L(G))$ is said to be dominating set of $L(G)$, if every vertex not in D is adjacent to a vertex in D of $L(G)$. A dominating set D is called pendant dominating set of $L(G)$, if the sub graph $\langle D \rangle$ contains at least one pendant vertex in $L(G)$. The minimum cardinality of vertices in such a set is called pendant domination number of $L(G)$ and is denoted by $\gamma_{pe}(L(G))$. In this paper, we obtain its exact values for some standard graphs. Also we establish its relationship with other parameters of graph. Nordhaus – Gaddum type result is also obtained.

KEY WORDS: Line graph, Pendant vertex, Dominating set , Pendant dominating Set, Pendant domination number. Subject classification number – MSCN:05C69.

1. INTRODUCTION

All the graphs considered here are simple, finite, non – trivial, undirected and connected.

As usual $p = |V|$ and $q = |E|$ denote the number of vertices and edges of G. In this paper, any undefined term can be found in Harary [1].

In general, we use $\langle X \rangle$ to denote the sub graph induced by the set of vertices X. $N(v)$ and $N[v]$ denote the open and closed neighborhoods of a vertex v, respectively. Let $\deg(v)$ is the degree of vertex v and as usual $\delta(G)$ ($\Delta(G)$) is the minimum (maximum) degree. The degree of an edge $e = uv$ is given by $\deg(e) = \deg(u) + \deg(v) - 2$. $\delta'(G)$ ($\Delta'(G)$) is the minimum (maximum) edge degree. A vertex of degree one is called an pendant vertex. The longest distance between any two vertices of a connected graph G is called the diameter of G, and is denoted $diam(G)$. A line graph $L(G)$ is the graph whose vertices corresponds to the edges of G and two vertices in $L(G)$ are adjacent if and only if the corresponding edges in G are adjacent (that is, are incident with a common vertex).

We begin by recalling some standard definitions from domination theory.

A set $S \subseteq V(G)$ is said to be a dominating set of G, if every vertex in $V - S$ is adjacent to some vertex in S. The minimum cardinality of vertices in such a set is called the domination number of G and is denoted by $\gamma(G)$. A set $D \subseteq V(L(G))$ is said to be dominating set of $L(G)$, if every vertex not in D is adjacent to a vertex in D of $L(G)$ .The domination number of $L(G)$ denoted by $\gamma(L(G))$ is the minimum cardinality of a dominating set in $L(G)$. The concept of dominate -on in graphs with its many variations is now well studied in graph theory, see [2, 3].

A dominating set D is called independent dominating set if $\langle D\rangle$ is also independent. The minimum cardinality of vertices in such a set is called independent domination in $L(G)$ and is denoted by $i(L(G))$. This concept was introduced by Muddebihal et. al., [4].

A dominating set S is called a pendant dominating set if the sub graph $\langle S\rangle$ contains at least one pendant vertex. The minimum cardinality of vertices in such a set is called pendant domination number and is denoted by $\gamma_{pe}(G)$. This concept was introduced by Nayaka et. al., [5].

Analogously, a dominating set D in $L(G)$ is called pendant dominating set of $L(G)$, if the sub graph $\langle D\rangle$ contains at least one pendant vertex. The minimum cardinality of vertices in such a set is called pendant domination number of $L(G)$ and is denoted by $\gamma_{pe}(L(G))$. In this paper, we obtain its exact values for some standard graphs. Its relationship with other parameters of graphs and line graphs were obtained. Also we give Nordhaus – Gaddum type result.

2. RESULTS

Initially we begin by giving exact values of pendant domination number of some standard graphs, which are straight forward.

Theorem 1:

i. For any path P_p with $p\geq 3$ vertices,

$$\gamma_{pe}(L(P_p)) = \frac{p}{3}+1, \text{ if } p\equiv 0(\text{mod}\,3)$$
$$= \left\lceil \frac{p}{3}\right\rceil, \text{ otherwise.}$$

ii. For any cycle C_p with $p\geq 3$ vertices,

$$\gamma_{pe}(L(C_p)) = \frac{p}{3}+1, \text{ if } p\equiv 0(\text{mod}\,3)$$
$$= \left\lceil \frac{p}{3}\right\rceil, \text{ if } p\equiv 1(\text{mod}\,3)$$
$$= \left\lceil \frac{p}{3}\right\rceil+1, \text{ if } p\equiv 2(\text{mod}\,3).$$

iii. For any star $K_{1,p}$ with p vertices,

$$\gamma_{pe}(L(K_{1,p}))=2.$$

iv. For any bipartite graph $K_{m,n}$,

$$\gamma_{pe}(L(K_{m,n}))=\min\{m,n\}+1.$$

v. For any wheel W_p with $p\geq 4$ vertices,

$$\gamma_{pe}(L(W_p)) = \frac{p}{3}+1, \text{ if } p\equiv 0(\text{mod}\,3)$$
$$= \left\lceil \frac{p}{3}\right\rceil, \text{ otherwise.}$$

Theorem 2: A dominating set $D\subseteq V(L(G))$ is a minimal pendant dominating set in $L(G)$ if and only if for each vertex $u\in D$, one of the following conditions hold:

i. u is either an isolate or pendant vertex of D in $L(G)$.

ii. each vertex of $D-\{u\}$ belongs to some cycle in $L(G)$.

iii. there exists a vertex $v\in V(L(G))-D$ for which $N(v)\cap D=\{u\}$.

Proof: Let $D=\{v_1,v_2,\dots,v_n\}\subseteq V(L(G))$ be a minimal pendant dominating set of $L(G)$. Assume that for every vertex $u\in D$, the set $D-\{u\}$ is not a pendant dominating set in $L(G)$. Then we have the following cases.

Case 1: $D-\{u\}$ is not a dominating set of $L(G)$. A set $D\subseteq V(L(G))$ is a minimal γ- set if for each vertex $u\in D$, either u is an isolated vertex of D in $L(G)$ or there exists a vertex $v\in V(L(G))-D$ such that $N(v)\cap D =\{u\}$.

Case 2: $D-\{u\}$ is a γ- set of $L(G)$, but contains no pendant vertex in $L(G)$. Here each vertex of $D-\{u\}$ is either an isolated vertex of D in $L(G)$ or has degree at least 2 in $L(G)$. If all the vertices of $D-\{u\}$ are isolated vertices of D, then u will be pendant vertex of D in $L(G)$. Further, if each vertex has degree at least two in $L(G)$, then each vertex of $D-\{u\}$ belongs to some cycle in $L(G)$.

Conversely, assume that D is a pendant dominating set which satisfies the three given conditions. On contrary, assume now that D is not a minimal pendant dominating set in $L(G)$. Then there exists a vertex $u \in D$ such that $D-\{u\}$ is also a minimal pendant dominating set in $L(G)$. Hence u must be adjacent to at least one vertex $v \in D-\{u\}$, so $\{u\}$ is not an isolate of D and if v is the pendant vertex of $D-\{u\}$, then v is not a pendant vertex of D, hence Condition (1) fails to hold. Clearly, Condition (2) does not holds since $D-\{u\}$ contains a pendant vertex. Finally every vertex in $V(L(G))-D$ must be adjacent to at least one vertex in $D-\{u\}$ of $L(G)$ so that the Condition (3) fails to hold. Hence it is a contradiction to our assumption that none of the above conditions holds. This proves that at least one of the conditions should hold.

Theorem 3: For any connected (p,q) - graph G with $p \geq 3$ vertices, the compliment $V(L(G))-D$ of any pendant dominating set is a dominating set if D contains no induced path with 3 vertices.

Proof: Let $D=(v_1,v_2,...,v_n) \subseteq V(L(G))$ be any pendant dominating set in $L(G)$. If the sub graph $\langle S \rangle$ contains no induced path with 3 vertices, then every vertex in $L(G)$ will be either a vertex of D adjacent to some vertex in D. Therefore, it follows that $V(L(G))-D$ forms a minimal dominating set in $L(G)$.

The following Theorem characterizes pendant domination and domination numbers of line graph.

Theorem 4: For any connected graph G, $\gamma_{pe}(L(G))=\gamma(L(G))$ if and only if

i. $L(G)$ contains a γ - set which is neither an independent set in $L(G)$ nor each vertex of D has degree zero or

ii. belongs to a cycle of D in $L(G)$.

Proof: For any graph G, if its line graph $L(G)$ is acyclic, then the result follows immediately. Further, assume now that $L(G)$ contains a cyclic graph and $\gamma_{pe}(L(G))= \gamma(L(G))$. Suppose every γ - set D in $L(G)$ is either independent or each vertex of D has degree zero or belongs to a cycle in D, then γ_{pe} - set of $L(G)$ is obtained by adding one vertex $u \in V(L(G))-D$ to a γ - set in $L(G)$. Hence $\gamma(L(G)) < \gamma_{pe}(L(G))$, a contradiction. Conversely, if every γ - set in $L(G)$ fails to satisfy the stated conditions, then the sub graph $\langle D \rangle$ must contain at least one pendant vertex. Therefore D itself is a pendant dominating set and so $\gamma_{pe}(L(G))=\gamma(L(G))$.

The following Theorem relates pendant domination and domination numbers of line graph interms of vertices of graph.

Theorem 5: For any connected (p,q) - graph with $p \geq 4$ vertices, $\gamma_{pe}(L(G))+\gamma(L(G)) \leq p$.

Proof: Let $F=\{v_1,v_2,...,v_n\}$ be the set of vertices with $\deg(v_i) \geq 2$, $\forall v_i \in F$, $1 \leq i \leq n$ in $L(G)$. Further, let $F^{'} \subseteq F$ such that $dist(u, \ v) \geq 2$. Then there exists a minimal set of verti –ces $F_1^{'} \subseteq F^{'}$ such that $N[F_1^{'}]=V(L(G))$. Clearly, $F_1^{'}$ forms a γ - set of $L(G)$. Now since each vertex of the sub graph $\langle F_1^{'} \rangle$ has degree zero, attaching a vertex w to any vertex of $F_1^{'}$ forms a pendant dominating set in $L(G)$. Otherwise, $D \subseteq N(F_1^{'}) \cup S$ where $S \subseteq V(L(G))-F_1^{'}$ gives a minimal pendant dominating set in $L(G)$. Clearly it follows that $|D| \cup |F_1^{'}| \leq |V(G)|$. Hence $\gamma_{pe}(L(G))+\gamma(L(G)) \leq p$.

The following Theorem relates pendant domination number of graph and line graph.

Theorem 6: For any connected (p,q) - graph G, except for K_p, $\gamma_{pe}(L(G))+\gamma_{pe}(G) \leq p \ -\delta(G)+2$.

Proof: Let $v \in V(G)$ such that $\deg(v)=\delta(G)$ and $S=\{v_1,v_2,...,v_n\} \subseteq V(G)$ bet he minimal set of vertices such that $N[v_i]=V(G)$, $1 \leq i \ \leq n$ and the sub graph $\langle S \rangle$ contains at least one pendant vertex, then S forms a minimal γ_{pe} - set of G. Now without loss of generality, in $L(G)$, let $F=\{u_1,u_2,...,u_k\}$ be the set of vertices corresponding to the edges incident to vertices of S in G. Now let $D \subseteq F$ such that $N[u_i]=V(L(G))$, $u_i \in D$ and if there exists at least one vertex $x \in D$ with $\deg(x)=1$ in the sub graph $\langle D \rangle$. Clearly, D forms a minimal pendant dominating set in $L(G)$. It follows that $|D| \cup |S| \leq |V(G)|-\delta(G)+2$. Hence $\gamma_{pe}(L(G))+\gamma_{pe}(G) \leq p-\delta(G)+2$.

The following Theorem relates pendant domination and independent domination numbers of line graph.

Theorem 7: For any connected (p,q) - graph G, $\gamma_{pe}(L(G)) \le i(L(G))+1$.

Proof: Let $C=\{v_1,v_2,...,v_n\}$ be the set of all non end vertices in $L(G)$. Further let there exists minimum set of vertices $F=C_1 \cup C^{'}$, where $C_1 \subseteq C$ and $C^{'} \subseteq V(L(G))-C$ such that $dist(x,y) \ge 2$, $\forall x,y \in F$ and if $N[F]= V(L(G))$. Clearly, F forms an i - set of $L(G)$. Now assume there exists a minimal set of vertices $D \subseteq V(L(G))$ which covers all the vertices in $L(G)$ and if the sub graph $\langle D \rangle$ contains at least one vertex $u \in D$ with $\deg(u)=1$. Then D itself is a pendant dominating set in $L(G)$. It follows that $|D| \le$ $|F| \cup 1$ and hence $\gamma_{pe}(L(G)) \le i(L(G))+1$.

In the following Theorem, we give lower bound for pendant domination number of $L(G)$ interms of edges and edge degree of a graph.

Theorem 8: For any connected (p,q) - graph G, $\left\lceil \frac{q}{\Delta^{'}(G)+1} \right\rceil \le \gamma_{pe}(L(G))$.

Proof: Let F be the set of vertices with $\deg(v_i) \ge 2$, $\forall v_i \in F$, $1 \le i \le n$ in $L(G)$. Further, let $D=\{v_1,v_2,...,v_k\} \subseteq F$ be the mini -mal set of vertices which covers all the vertices in $L(G)$. Suppose the sub graph $\langle D \rangle$ contains at least one vertex v with $\deg(v)=1$, then D itself is a pendant dominating set in $L(G)$. Otherwise, attach a vertex $x \in V \;\; (L(G))-D$ to a vertex of D. Then $D \cup \{x\}$ forms a minimal γ_{pe} - set in $L(G)$. Since for any connected graph G, there exists at least one edge $e \in E(G)$ with $\deg(e)=\Delta^{'}(G)$, we have $\left\lceil \frac{|E(G)|}{\deg(u) \cup 1} \right\rceil \le |D \cup \{x\}|$ .Hence it follows $\left\lceil \frac{q}{\Delta^{'}(G)+1} \right\rceil \le \gamma_{pe}(L(G))$.

In the following Theorem, we give upper bound for pendant domination number of line graph of tree interms of vertices of tree.

Theorem 9: If every non end vertex of a tree is adjacent to at least one end vertex, then $\gamma_{pe}(L(T)) \le n-1$, where n is the number of end vertices.

Proof: Let $A=\{v_1,v_2,...,v_n\}$ be the set of all end vertices in T with $|A|=n$. Now by definit -ion of line graph, let $F=\{u_1,u_2,...,u_k\} \subseteq V \;\; (L(T))$ be the set of vertices corresponding to the edges which are incident with the vertices of A in T. Suppose there exists a set $D \subseteq F$ such that $N[D]=V(L(T))$ and if D has at least one vertex u with $\deg(u)=1$. Then D forms a minimal pendant dominating set in $L(T)$. Else attach a vertex w to the vertex of D such that $D \cup \{w\}$ forms a minimal γ_{pe} - set of $L(T)$. It follows that $|D \cup \{w\}| \le |A|-1$. Hence $\gamma_{pe}(L(T)) \le n-1$.

Finally, we give Nordhaus – Gaddum type result.

Theorem 10: For any connected (p,q) - graph G:

i. $\gamma_{pe}(L(G))+\gamma_{pe}(L(\bar{G})) \le 2\left(\left\lfloor \frac{q}{2} \right\rfloor+1\right)$.

ii. $\gamma_{pe}(L(G)) \cdot \gamma_{pe}(L(\bar{G})) \le 2\left(\left\lfloor \frac{q}{2} \right\rfloor+1\right)^2+q+1$.

Conflict of interest statement

Authors declare that they do not have any conflict of interest.

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