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A Detailed Proof of Rolle's Theorem

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ABSTRACT

The aim of this paper is to remove the difficulties in understanding the proof of Rolle's theorem by the students. To achieve this we used the definition of limit and trichotomy law ofreal numbers.

KEY WORDS: Rolle's Theorem, Trichotomy, Real Numbers.

INTRODUCTION

The proof of Rolle's Theorem will be given in many text books of "Real Analysis". Vishal V.Mehtre and Shubham gupta[1] did survey on understanding the proof of Rolle's theorem by the students. They observed many students have difficulty in understanding the proof of Rolle's Theorem. Perhaps this may be due to the use of certain results which were proved earlier in the proof of Rolle's Theorem. The students may not recollect these results at the time hearing the proof of it. Further the students have doubt in certain conclusions used in the proof of Rolle's Theorem. In this we are presenting the detailed proof of Rolle's theorem using the definition of limit and trichotomy law of real numbers to remove the doubts of the students and easy understanding the proof of it.

Statement of Rolle's Theorem:

Let $a, b \in R, a < b, \quad I = [a,b]$ and if

$i)$ $f(x)$ is continous on the closed interval I

$ii)$ $f(x)$ is differentiable on the open interval (a,b)

iii) $f(a) = f(b)$

then there exists at least one real number $c \in (a,b)$ with $f'(c) = 0$

Proof of Rolles Theorem :

By the hypothesis the function $f(x)$ is continuous on the closed interval $[a$,b]. Therefore, by the Boundedness theorem [2] the function $f(x)$ attains absolute maximum and absolute minimum on $[a,b]$.

Let M and m be absolute maximum and absolute minimum of $f(x)$ on $[a,b]$ respectively. Then there exist points $c \in I$ and $d \in I$. Such that

$$f(c) = M \quad (1)$$

And

$$f(d) = m \quad (2)$$

Now there are two possibilities $M = m$ and $M \neq m$.

Case (i) : $M = m$

Since M and m are absolute maximum and absolute minimum of $f(x)$ on $[a,b]$. We have

$$m \leq f(x) \leq m \ \forall \ x \in [a,b] \quad (3)$$

If $M = m$ then $m \leq f(x) \leq m \ \forall \ x \in [a,b]$.

$m \leq f(x) \leq m \ \forall \ x \in [a,b]$

$\Rightarrow f(x) = m \ \forall \ x \in [a,b]$ (4)

From (4) we can conclude that the function $f(x)$ is constant function on $[a,b]$. Hence $f'(x) = 0 \ \forall \ x \in [a,b]$.

The conclusion of the theorem is proved in this case.

Case (ii) $M \neq m$

First we prove that $c \in (a,b)$.

Since $M \neq m$, one M and m cannot be equal to $f(a)$.

Let $M \neq f(a)$ (5)

From (1) and (5) we obtain $f(c) \neq f(a)$ (6)

By the hypothesis $f(a) = f(b)$ (7)

From (6) and (7) we obtain $f(c) \neq f(b)$ (8)

From (6) and (8) we have $c \neq a$ and $c \neq b$

Thus $c \in (a,b)$ .Clearly c lies in the open interval (a,b).

Now we prove that $f'(c) = 0$.The derivative of $f(x)$ exists at $x = c$.

Since $f(x)$ is differentiable in the open interval and .Hence exists.

Now there are three possibilities for two real numbers and

By trichonomy law of real numbers.

(p_1) $f'(c) > 0$ (p_2) $f'(c) < 0$

(p_3) $f'(c) = 0$

The conclusion of the proof follows if we show that $f'(c)$ not greater than 0 and $f'(c)$ not less than 0.

(i) Consider the possibility P_1**:**

Suppose $f'(c) > 0$

$$f'(c) > 0 \Rightarrow \lim_{x \to c} \frac{f(x) - f(c)}{x - c} = f'(c) > 0 \quad (9)$$

By the definition of limit for $\varepsilon = \frac{f'(c)}{2} > 0$ $\exists$ a number $\delta > 0$ such that

$$\left| \frac{f(x) - f(c)}{x - c} - f'(c) \right| < \frac{f'(c)}{2} \quad \forall \ x \in [(c - \delta, c + \delta) - \{c\}]$$

(10)

$$\Rightarrow \frac{f'(c)}{2} < \frac{f(x) - f(c)}{x - c} < \frac{3}{2} f'(c) \ \forall \ x \in [(c - \delta, c + \delta) - \{c\}] \quad 11)$$

From (11) we have

$$\frac{f(x) - f(c)}{x - c} > \frac{f'(c)}{2} \ \forall \ x \in [(c - \delta, c + \delta) - \{c\}] \quad (12)$$

Since $f'(c) > 0$ we have $\frac{f(x) - f(c)}{x - c} > 0$

(13)

If $x > c$ then

$$f(x) - f(c) = \frac{f(x) - f(c)}{x - c} \times (x - c) > 0 \quad \forall \ c < x < c + \delta$$

If $x > c$ then $f(x) - f(c) > 0 \quad \forall \ c < x < c + \delta$

If $x > c$ then $f(x) > f(c) \quad \forall \ c < x < c + \delta$

It is a contradiction to the fact of $f(c)$ is absolute maximum.Therefore $f'(c)$ is cannot greater than 0.Thus the possibility (P_1) cannot hold.

b) Consider the possibility P_2 **:**

Let $f'(c) < 0$

(14)

$$\lim_{x \to c} \frac{f(x) - f(c)}{x - c} = f'(c)$$

(15)

By the definition of limit for $\varepsilon = -\frac{f'(c)}{2} > 0$ $\exists$ a number $\delta > 0$ $\exists$

$$\left| \frac{f(x) - f(c)}{x - c} - f'(c) \right| < -\frac{f'(c)}{2} \quad \forall \ x \in [(c - \delta, c + \delta) - \{c\}] \quad 16)$$

$$\Rightarrow \frac{f(x) - f(c)}{x - c} < \frac{f'(c)}{2} \ \forall \ x \in [(c - \delta, c + \delta) - \{c\}]$$

(17)

Since $f'(c) < 0$ we have

$$\frac{f(x) - f(c)}{x - c} < 0 \quad \forall \ x \in [(c - \delta, c + \delta) - \{c\}]$$

(18)

If $x < c$ then

$$f(x) - f(c) = \frac{f(x) - f(c)}{x - c} \times (x - c) > 0 \ \forall \ c - \delta < x < c$$

$\Rightarrow$ If $x < c$ then $f(x) - f(c) > 0 \ \forall \ x \in (c - \delta, c)$

$\Rightarrow$ If $x < c$ then $f(x) > f(c) \ \forall \ x \in (c - \delta, c)$

It is a contradiction to the fact of $f(c)$ is absolute maximum. Therefore $f'(c)$ is not less than 0.We have shown that $f^{'}(c)$ is not greater than 0 and it is not less than 0.

Therefore by trichotomy law of real numbers we must have $f^{'}(c)=0$.

Conflict of interest statement

Authors declare that they do not have any conflict of interest.

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