



# Bayesian and Classical Methodologies for Reliability Analysis and Parameter Estimation of Weibull Extension Model under Informative Set of Priors

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## ABSTRACT

In this paper, we consider two methodologies for the parameter estimation and reliability analysis of the Weibull Extension model. In particular, the Maximum Likelihood Estimator of the shape and scale parameter is introduced. We then derived the Bayes estimators for the shape and scale parameter by considering gamma priors. The proposed estimators have been obtained using the Markov chain Monte Carlo (MCMC) method in OpenBUGS. Finally, the real data set has been used for illustrations to show the performance of the estimation and to establish a comparison between these two proposed methodologies.

**KEYWORDS:** Bayesian Methodologies, Reliability analysis, Parameter Estimation, Weibull Extension Model, Informative set of Prior, MCMC, R Function, OpenBUGS.

## I. INTRODUCTION

Weibull distribution is one of the most commonly used life-time distributions in reliability and life time data analysis. It is flexible in modeling failure time data, as the corresponding hazard rate function can be increasing, constant or decreasing. But in many applications in reliability and survival analysis, the hazard rate function can be of bathtub shape. The bathtub-shape hazard function provides an appropriate conceptual model for some electronic and mechanical products as well as the lifetime of humans. Models with a bathtub hazard rate function are needed in reliability analysis and decision making when the life time of the system is to be modeled.

Many parametric probability distributions have been introduced to analyze sets of real data with

bathtub-shaped hazard rates [1]. The bathtub-shaped hazard function provides an appropriate conceptual model for some electronic and mechanical products as well as the life time of humans. Though modified Weibull distribution and Weibull extension are mentioned to have bathtub-shaped failure rate, they may not be able to give a good bathtub shape of the failure rate [2]. Zhang et al. [3] discussed the parametric analysis of some models which exhibit a good bathtub shape. A two-parameter Weibull Extension model was retraced by Chen (2000) [4]. This model can have a bathtub-shaped or increasing failure rate function which enables it to fit real lifetime data sets [5].

In this paper, we have presented the two-parameter Weibull Extension model as

software reliability model which is motivated by the fact that the existing models are inadequate to describe the failure process underlying some of the data sets.

The cumulative distribution function of the distribution is

$$F(x; \beta, \lambda) = 1 - \exp\left\{\lambda \left\{1 - \exp\left(x^\beta\right)\right\}\right\}; \quad (1)$$

where  $(\beta, \lambda) > 0, x > 0$

here  $\lambda > 0$  is a parameter which does not affect the shape of the failure rate function and  $\beta > 0$  named as shape parameter.

## II. ANALYSIS OF MATHEMATICAL PROPERTIES

The Weibull Extension model has the probability density function

$$f(x; \beta, \lambda) = \beta \lambda x^{\beta-1} \exp\left(x^\beta\right) \exp\left\{\lambda \left\{1 - e^{x^\beta}\right\}\right\}; \quad (2)$$

where,  $(\beta, \lambda) > 0, x > 0$

The Weibull Extension model has the probability density function given as equation (2) on a support of  $[0, \infty)$ . Hence, there could be three types of shape of the density function. The shapes of density function are examined by looking at the first derivative of the density function. The shapes of the density function can be classified as follows:

### Case 1: $\lambda \geq 1$

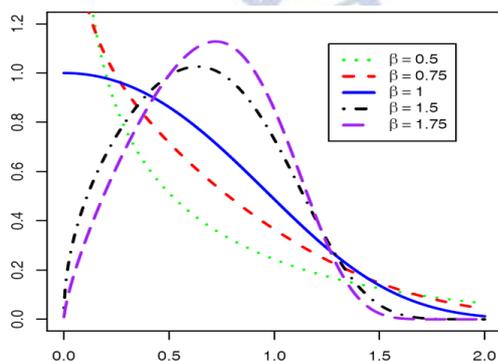
- (i) If  $\beta > 1$ ,  $f(x)$  is unimodal; and
- (ii) If  $0 < \beta \leq 1$ ,  $f(x)$  is a decreasing function.

### Case 2: $\lambda < 1$

- (i) If  $\beta < 1$ ,  $f(x)$  is unimodal;
- (ii) If  $\beta = 1$ ,  $f(x)$  is a decreasing function; and
- (iii) If  $0 < \beta < 1$ ,  $f(x)$  is either a decreasing function or a function that first decreases, then increases and finally decreases again.

The shape properties of the tails of the density function can be classified as follows.

- (i)  $\beta < 1$ ,  $f(x)$  is unimodal shaped and  $f(x) \rightarrow 0$  when  $x \rightarrow 0$ .
- (ii)  $\beta = 1$ ,  $f(x) \rightarrow \lambda$  when  $x \rightarrow 0$ , hence the density function is high tail at the left hand.
- (iii)  $0 < \beta < 1$ ,  $f(x) \rightarrow \infty$  when  $x \rightarrow 0$ , the distribution has an unbounded high left tail.



**Figure 1:** PDF Plots Weibull Extension model for  $\lambda=1$  and different values of  $\beta$

Our developed R functions  $dwem\_aks()$  and  $pwem\_aks()$  can be used for the computation of pdf and cdf, respectively. Some of the typical Weibull Extension density functions for different values of  $\beta$  and for  $\lambda = 1$  are depicted in Figure 1 which also clear that the density function of the Weibull Extension model can take different shapes.

For a continuous distribution  $F(x)$ , the  $p$  percentile (also referred to as fractile or quantile),  $x_p$ , for a given  $p, 0 < p < 1$ , is a number such that

$$P(X \leq x_p) = F(x_p) = p. \quad (3)$$

The quantile for  $p=0.25$  and  $0.75$  are called first and third quartiles and the  $0.50$  quantile is called the median ( $Q_2$ ). The five parameters

*minimum(x),  $Q_1, Q_2, Q_3,$  maximum(x)*

are often referred to as the **five-number summary or explanatory data analysis**. Together, these parameters give a great deal of information about the distribution in terms of the centre, spread, and skewness. Graphically, the five numbers are often displayed as a **boxplot**. **The quantile function of WEM( $\beta, \lambda$ ) can be obtained by solving**

$$x_p = \left[ \ln \left\{ 1 - \frac{\ln(1-p)}{\lambda} \right\} \right]^{1/\beta}; \quad 0 < p < 1. \quad (4)$$

The computation of quantiles, our developed R function  $qwem\_aks()$  can be used. In particular, for  $p=0.5$  we get

$$\text{Median}(x_{0.5}) = \left[ \ln \left\{ 1 - \frac{\ln(0.5)}{\lambda} \right\} \right]^{1/\beta} \quad (5)$$

The mode can be obtained by solving the following non-linear equation

$$(\beta - 1) + \beta x^\beta \left\{ 1 - \lambda \exp\left(x^\beta\right) \right\} = 0 \quad (6)$$

Let  $U$  be the uniform  $(0,1)$  random variable and  $F(\cdot)$  a cdf for which  $F^{-1}(\cdot)$  exists. Then  $F^{-1}(u)$  is a draw from distribution  $F(\cdot)$ . Therefore, the random deviate can be generated from WEM( $\beta, \lambda$ ) by

$$x = \left[ \ln \left\{ 1 - \frac{\ln(1-u)}{\lambda} \right\} \right]^{1/\beta}; \quad 0 < u < 1. \quad (7)$$

where  $u \sim U(0, 1)$ . Our developed R function  $rwem\_aks()$  generates the random deviate from WEM( $\beta, \lambda$ ).

The reliability/survival function of WEM( $\beta, \lambda$ )

$$R(x; \beta, \lambda) = \exp\left\{\lambda \left\{1 - \exp\left(x^\beta\right)\right\}\right\}; \quad (\beta, \lambda) > 0, x > 0 \quad (8)$$

Our developed R function  $swem\_aks()$  computes the reliability/ survival function.

The hazard function of WEM( $\beta, \lambda$ ) is given by

$$h(x) = \beta \lambda x^{\beta-1} \exp(x^\beta) ; (x > 0, \beta > 0, \lambda > 0) \quad (9)$$

and, the computation of hazard function of WEM( $\beta, \lambda$ ) by developed R function *hwem\_aks()*.

Differentiating equation (9) w.r.t  $x$ , we have

$$h'(x) = \beta \lambda x^{\beta-2} \exp(x^\beta) \{(\beta-1) + \beta x^\beta\} ; (x > 0) \quad (10)$$

The shape of hazard function depends only on the shape parameter  $\beta$ . For  $\beta \geq 1$

- (i)  $h(x)$  is an increasing function.
- (ii)  $h(0) = 0$  if  $\beta > 1$  and  $h(0) = 1$ , if  $\beta = 1$ .
- (iii)  $h(x) \rightarrow +\infty$  as  $x \rightarrow +\infty$ .

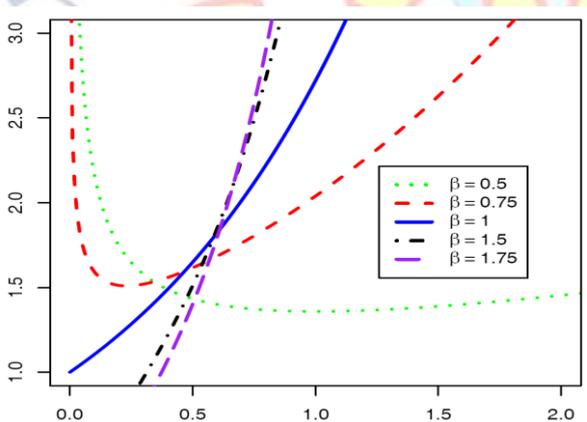
For  $0 < \beta < 1$

- (i)  $h(x)$  is decreasing for  $x^* < x$  and increasing for  $x^* > x$  where

$$x^* = \left(\frac{1}{\beta} - 1\right)^{1/\beta} \quad (11)$$

This implies that the hazard function has a bathtub type shape.

- (ii)  $h(x) \rightarrow +\infty$  as  $x \rightarrow 0$  or  $x \rightarrow +\infty$ .
- (iii)  $x^*$  increases as the shape parameter  $\beta$  decreases.



**Figure 2:** The hazard function Plots of Weibull Extension model for  $\lambda=1$  and different values of  $\beta$

Some of the typical hazard functions of WEM( $\beta, \lambda$ ) for different values of  $\beta$  and for  $\lambda=1$  are depicted in Figure 2. It is clear from the Figure 2 that the hazard function of the WEM( $\beta, \lambda$ ) can take different shapes.

The cumulative hazard function  $H(x)$  of WEM( $\beta, \lambda$ ) defined as

$$H(x) = -\{1 - \log F(x)\} \quad (12)$$

can be obtained with the help of *pwem\_aks()* function by choosing arguments *lower.tail = FALSE* and *log.p = TRUE*. i.e.

- *pwem\_aks(x, beta, lambda, lower.tail = FALSE, log.p = TRUE)*

In WEM( $\beta, \lambda$ ), the failure rate average of  $X$  is given by

$$FRA(x) = \frac{H(x)}{x} = \frac{1}{x} \int_{x_0}^x h(x) dx, \quad x > 0 \quad (13)$$

where  $H(x)$  is the cumulative hazard function. An analysis for  $FRA(x)$  on  $x$  permits to obtain the IFRA and DFRA classes.

The conditional survival of  $X$  are defined by

$$R(x|t) = \frac{R(x+t)}{R(x)}, \quad t > 0, x > 0, R(\bullet) > 0 \quad (14)$$

respectively, where  $F(\cdot)$  is the cdf of  $X$ . Our developed R functions *hra.wem\_aks()* and *crf.wem\_aks()* can be used for the computation of failure rate average (fra) and conditional survival function (crf), respectively

### III. MAXIMUM LIKELIHOOD ESTIMATION FOR PARAMETERS OF WEM( $\beta, \lambda$ )

In this section we use the Maximum Likelihood method to estimate the two unknown parameters  $\beta$  and  $\lambda$  [6]. Let us suppose that  $x=(x_1, \dots, x_n)$  be a random sample from WEM( $\beta, \lambda$ ) with cumulative distribution function (1). The log likelihood function of the parameter  $L(\beta, \lambda)$  is given by

$$L = (\beta \lambda)^n \left[ \prod_{i=1}^n x_i \right]^{\beta-1} e^{-\sum_{i=1}^n \left\{ x_i^\beta + \lambda \left( 1 - e^{-x_i^\beta} \right) \right\}} \quad (15)$$

or,

$$L(\beta, \lambda) = \beta^n \lambda^n \left( \prod_{i=1}^n x_i^{\beta-1} \right) \exp \left( \sum_{i=1}^n x_i^\beta \right) \exp \left[ \lambda \sum_{i=1}^n \left\{ 1 - \exp \left( -x_i^\beta \right) \right\} \right] \quad (16)$$

The log-likelihood function becomes

$$L = n(\ln \beta + \ln \lambda) + (\beta-1) \sum_{i=1}^n \ln x_i + \sum_{i=1}^n \left\{ x_i^\beta + \lambda \left( 1 - e^{-x_i^\beta} \right) \right\} \quad (17)$$

Therefore, to obtain the MLE's of  $\beta$  and  $\lambda$  we can maximize (17) directly with respect to  $\beta$  and  $\lambda$  or we can solve the following two non-linear equations using iterative procedure [5]:

$$\frac{\partial \log L}{\partial \lambda} = \frac{n}{\lambda} + \sum_{i=1}^n \left\{ 1 - \exp \left( -x_i^\beta \right) \right\} = 0 \quad (18)$$

$$\frac{\partial \log L}{\partial \beta} = \frac{n}{\beta} + \sum_{i=1}^n \log x_i + \beta \sum_{i=1}^n x_i^{\beta-1} (\log x_i) + \lambda \beta \sum_{i=1}^n x_i^{\beta-1} \exp \left( -x_i^\beta \right) = 0 \quad (19)$$

Since the MLEs of the unknown parameters  $L(\beta, \lambda)$  can not be obtained in closed forms, it is not easy to derive the exact distributions of the MLEs. We can derive the asymptotic confidence intervals of these parameters when  $\beta > 0$  and  $\lambda = 0$ . The simplest large sample approach is to assume that the MLE  $(\hat{\beta}, \hat{\lambda})$  are approximately bivariate normal with mean  $(\beta, \lambda)$  and covariance matrix  $I_0^{-1}$ , [7], where  $I_0^{-1}$  is the inverse of the observed information matrix

$$I_0^{-1} = \begin{pmatrix} -\frac{\partial^2 \ln L}{\partial \beta^2} \Big|_{\hat{\beta}, \hat{\lambda}} & -\frac{\partial^2 \ln L}{\partial \beta \partial \lambda} \Big|_{\hat{\beta}, \hat{\lambda}} \\ -\frac{\partial^2 \ln L}{\partial \lambda \partial \beta} \Big|_{\hat{\beta}, \hat{\lambda}} & -\frac{\partial^2 \ln L}{\partial \lambda^2} \Big|_{\hat{\beta}, \hat{\lambda}} \end{pmatrix}^{-1} = \left( -H \Big|_{(\hat{\beta}, \hat{\lambda})} \right)^{-1} \\ = \begin{pmatrix} \text{var}(\hat{\beta}) & \text{cov}(\hat{\beta}, \hat{\lambda}) \\ \text{cov}(\hat{\beta}, \hat{\lambda}) & \text{var}(\hat{\lambda}) \end{pmatrix}. \quad (20)$$

The above approach is used to derive the 100(1- $\gamma$ /2)% confidence intervals of the parameters  $L(\beta, \lambda)$  as in the following forms

$$\hat{\beta} \pm z_{\gamma/2} \sqrt{\text{Var}(\hat{\beta})} \quad \text{and} \quad \hat{\lambda} \pm z_{\gamma/2} \sqrt{\text{Var}(\hat{\lambda})} \quad (21)$$

Here,  $Z_{\gamma/2}$  is the upper ( $\gamma/2$ )th percentile of the standard normal distribution.

#### IV. BAYESIAN ESTIMATION FOR PARAMETERS OF WEM( $\beta, \lambda$ )

This section presents Bayes estimates of the parameters  $\lambda$  and  $\beta$ . It is assumed here that the parameters  $\lambda$  and  $\beta$  are independent and follow the gamma prior distributions [8].

$$\pi_1(\beta) \propto \beta^{a_1-1} \exp(-a_2\beta), \quad \beta > 0 \quad (22)$$

$$\pi_2(\lambda) \propto \lambda^{b_1-1} \exp(-b_2\lambda), \quad \lambda > 0 \quad (23)$$

where all the hyperparameters  $a_1$ ;  $a_2$ ;  $b_1$  and  $b_2$  are assumed to be nonnegative and known.

Using (22), (23), (17), and applying Bayes theorem, the posterior joint pdf of (a,b), for a; b > 0, occurs

$$\pi_2(\beta, \lambda | \text{data}) = \frac{1}{I_{(0,0)}} \beta^{n+a_1-1} \lambda^{n+b_1-1} P_x^{\beta-1} \exp\left(-\{a_2 + K_2(\beta)\}\lambda - b_2\beta + K_1(\beta)\right), \quad (24)$$

Where,

$$I_{(0,0)} = \iint \beta^{n+a_1-1} \lambda^{n+b_1-1} P_x^{\beta-1} \exp\left(-\{a_2 + K_2(\beta)\}\lambda - b_2\beta + K_1(\beta)\right) d\beta d\lambda,$$

$$\text{and, } P_x = \prod_{i=1}^n x_i, \quad K_1(\beta) = \sum_{i=1}^n x_{\beta}^i, \quad K_2(\beta) = \sum_{i=1}^n \exp(x_{\beta}^i) - 1.$$

For  $\beta > 0, \lambda > 0$ , the marginal posterior pdf's of  $\beta$  and  $\lambda$  can be derived from (24), we have

$$\pi_{\lambda}(\lambda | \text{data}) = \frac{\lambda^{n+a_1-1} \exp(-a_2\lambda)}{I_{(0,0)}} \quad (25)$$

$$\int_0^{\infty} P_x^{\beta-1} \beta^{n+b_1-1} \exp(-b_2\beta + K_1(\beta) - K_2(\beta)\lambda) d\beta \\ \pi_{\beta}(\beta | \text{data}) = \frac{1}{I_{(0,0)}} \frac{\Gamma(n+a_1)}{[a_2 + K_2(\beta)]^{n+a_1}} P_x^{\beta-1} \beta^{n+b_1-1} \exp(-b_2\beta + K_1(\beta)) \quad (26)$$

Now, we use approximation technique to obtain Bayes Estimates. There are many approximation techniques are used for this purpose. Here we use Monte Carlo integration method.

Let us consider a complex integral,

$$I = \int_a^b m(x) dx \quad (27)$$

If the function  $m(x)$  can be broken into the product of a function  $n(x)$  and a probability density function  $p(x)$  defined over the support (a,b), then the integral  $I$  can be expressed as the expectation of  $n(x)$  with respect to the probability density function  $p(x)$

$$I = \int_a^b n(x)p(x) dx = E_{p(x)}[n(x)], \quad (28)$$

where the subscript  $p(x)$  means that the expectation is taken with respect to  $p(x)$ . If we generate a large random sample  $x_1; x_2; \dots; x_M$  from the density  $p(x)$ , then

$$I = E_{p(x)}[n(x)] \approx \frac{1}{M} \sum_{i=1}^M n(x_i) \quad (29)$$

is an unbiased estimator of  $I$ . This is referred to as Monte Carlo integration [9].

It is rare that samples from a posterior distribution can be taken directly. A Markov chain is constructed instead of generating an independent sample from the posterior, then using Monte Carlo idea we can solve the problem [10].

A Markov chain is a sequence of realizations of a random variable, and the probability of drawing a new value only depends on the realization of the one immediately preceding. It does not depend on the entire past history. MCMC is essentially *Monte Carlo integration using Markov chains*. The integration is done by generating the samples from the required distribution, and then compute the sample averages to approximate expectations [11].

The MCMC methods have been shown to be easy to implement computationally, the estimates always exist and their probability intervals are convenient to construct [12].

Assuming that we have constructed a Markov chain, using MCMC algorithm with desirable properties, we then

1. Select an initial value  $\theta^{(0)}$ .
2. Generate T values until the equilibrium distribution is reached.
3. Monitor the convergence of the algorithm using convergence diagnostics. If convergence diagnostics fail, we then generate more observations.
4. Cut off the first B observations.
5. Consider  $\{\theta^{(B+1)}, \theta^{(B+2)}, \dots, \theta^{(T+1)}\}$  as the sample for the posterior analysis.
6. Plot the posterior distribution (usually focus is on the univariate marginal distributions).
7. Finally, obtain summaries of the posterior distribution (mean, median, standard deviation, quantiles, and correlations).

In these steps, we refer to *convergence diagnostics*, which are statistical tests that attempt to identify cases where convergence is not achieved.

## V. DATA ANALYSIS AND NUMERICAL OUTCOME

In this section, we are using software reliability data set *SYS2.DAT* - 86 time-between-failures [13] is considered for illustration of the proposed methodology. In this real data set, Time-between-failures is converted to time to failures and scaled [14]. By using this real data set, our analysis is in this manner:

- A. Here, first we compute the Maximum likelihood (ML) estimation of parameters of  $WEM(\beta, \lambda)$ , we use this real data set for illustration of the proposed methodology.
- B. To check the validity of the model, i.e., The proposed model is used to fit this data set.
- C. Bayesian Estimation of Parameters of  $WEM(\beta, \lambda)$
- D. Comparison with MLEs and Bayesian estimates

### MLE of Parameters for $WEM(\beta, \lambda)$ model

We have started the iterative procedure by maximizing the log-likelihood function given in (17) directly with an initial guess for  $\beta = 0.1$  and  $\lambda = 0.2$ , far away from the solution. We have used *optim()* function in R with option Newton-Raphson method. The iterative process stopped only after 5 iterations. We obtain  $\hat{\alpha} = 0.23760325$ ,  $\hat{\lambda} = 0.02211914$  and the corresponding log-likelihood value = -563.2257. The similar results are obtained using *maxLik* package available in R [15]. An estimate of variance-covariance matrix, using (20), is given by

$$\begin{pmatrix} \text{var}(\hat{\beta}) & \text{cov}(\hat{\beta}, \hat{\lambda}) \\ \text{cov}(\hat{\beta}, \hat{\lambda}) & \text{var}(\hat{\lambda}) \end{pmatrix} = \begin{bmatrix} 9.045893e-05 & -5.332551e-05 \\ -5.332551e-05 & 3.712440e-05 \end{bmatrix}$$

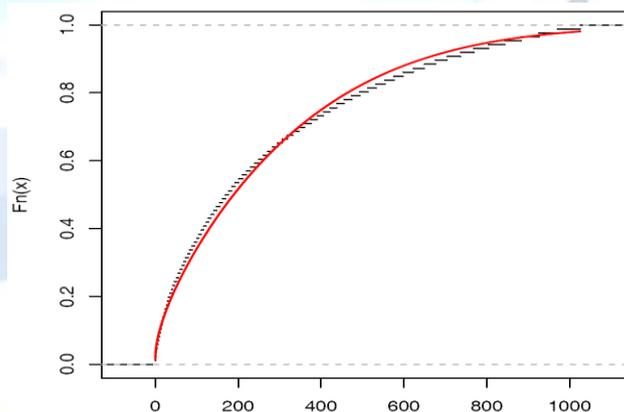
Thus using (21), we can construct the approximate 95% confidence intervals for the parameters of  $WEM(\beta, \lambda)$  based on MLE's. Table 1 shows the MLE's with their standard errors and approximate 95% confidence intervals for  $\beta$  and  $\lambda$ .

Parameter	MLE	Std. Error	95% Confidence Interval
beta	0.237603	0.009511	(0.218961, 0.256244)
lambda	0.022119	0.006093	(0.010176, 0.034061)

**Table 1:** MLE, standard error and 95% confidence interval

### Model validation of $WEM(\beta, \lambda)$ model

To check the validity of the model we compute the Kolmogorov-Smirnov (KS) distance between the empirical distribution function and the fitted distribution function when the parameters are obtained by method of MLE [16]. The result of K-S test is  $D=0.0342$ , with the corresponding p-value=0.9999. We have used the self-developed R function *ks.gen.wem\_aks()* is used for computation. We have plotted the empirical distribution function and the fitted distribution function in Figure 3, which shows very clear that the estimated  $WEM(\beta, \lambda)$  model provides *excellent fit* to the given data.



**Figure 3:** The graph of empirical and fitted model function.

### Bayesian Estimation of Parameters for $WEM(\beta, \lambda)$

We develop a module *aks.dwem(beta, lambda)* in open source software package OpenBUGS and implemented to obtain the Bayes estimates of the  $WEM(\beta, \lambda)$  model using MCMC method [17]. The

main function of the module is to generate MCMC sample from posterior distribution for Gamma prior [18].

The most widely used prior distribution of  $\theta$  is the inverted gamma distribution with parameters  $(a, b > 0)$  with p.d.f. given by

$$\pi(\theta) = \begin{cases} \frac{b^a}{\Gamma(a)} \theta^{a-1} e^{-b\theta} & ; \theta > 0 \quad (a, b) > 0 \\ 0 & ; \text{otherwise} \end{cases}$$

A module *aks.dwem(beta, lambda)* is written in language Component Pascal enables to perform full Bayesian analysis of *WEM*  $(\beta, \lambda)$  model into OpenBUGS using the method described in [19]. The code contains three parts:

*model, data, and initial values.*

Computer Code for the above module is as follows:

```

model
{
  for( i in 1 : N )
  {
    x[i] ~ dchen_T(beta, lambda)
  }
  beta ~ dgamma(0.001, 0.001)
  lambda ~ dgamma(0.001, 0.001) }

```

#### **Data**

```
list(N=86, x=c(0.05, .....1025.94))
```

#### **Initial values**

##### **# Chain1**

```
list(beta=0.2, lambda=0.02)
```

##### **# Chain2**

```
list(beta=0.8, lambda=0.10)
```

We run the model to generate two Markov Chains at the length of 40,000 with different starting points of the parameters. The convergence is monitored using trace and ergodic mean plots, we find that the Markov Chain converge together after approximately 2000 observations. Therefore, burn-in of 5000 samples is more than enough to erase the effect of starting point(initial values). Finally, samples of size 7000 are formed from the posterior by picking up equally spaced every fifth outcome, i.e. thin=5, starting from 5001. This are done to minimize the auto correlation among the generated deviates [20].

Therefore, we have the posterior sample  $\{\beta_{1i}, \lambda_{1i}\}$ ,  $i = 1, \dots, 7000$  from chain 1 and  $\{\beta_{2i}, \lambda_{2i}\}$ ,  $i = 1, \dots, 7000$  from chain 2.

The chain 1 is considered for convergence diagnostics plots. The visual summary is based on

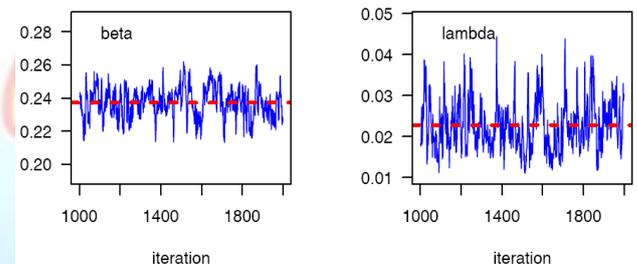
posterior sample obtained from chain 2 whereas the numerical summary is presented for both the chains.

#### **a. Convergence diagnostics**

Before examining the parameter estimates or performing other inference, it is a good idea to look at plots of the sequential (dependent) realizations of the parameter estimates and plots thereof. We have found that if the Markov chain is not mixing well or is not sampling from the stationary distribution, this is usually apparent in sequential plots of one or more realizations. This is test out by these plots:

##### **• History(Trace) plot**

The sequential plot of parameters is the plot that most often exhibits difficulties in the Markov chain. Figure 4 shows the sequential realizations of the parameters of the model. In this case Markov chain seems to be mixing well enough and is likely to be sampling from the stationary distribution.



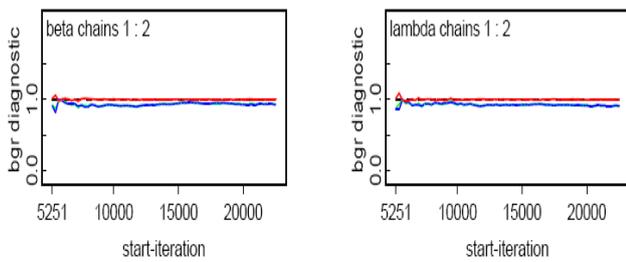
**Figure 4:** Seq. realization of the parameters beta and lambda

The plot looks like a horizontal band, with no long upward or downward trends, then we have evidence that the chain has converged.

##### **• Brooks-Gelman-Rubin(BGR) diagnostic**

The Brooks, Gelman and Rubin convergence diagnostic is appropriate for the analysis of two or more parallel chains, each with different starting values which are overdispersed with respect to the target distribution [21].

OpenBUGS provides the Brooks-Gelman-Rubin statistic for assessing convergence. For a given parameter, this statistic assesses the variability within parallel chains as compared to variability between parallel chains. The model is judged to have converged if the ratio of between to within variability is close to 1. Plots of this statistic can be obtained in OpenBUGS by clicking the “bgr diag” (Brook, Gelman & Rubin) button. The green line represents the between variability, the blue line represents the within variability, and the red line represents the ratio.



**Figure 5:** The BGR plots for beta and lambda

Evidence for convergence comes from the red line being close to 1 on the y-axis and from the blue and green lines being stable (horizontal) across the width of the plot.

From the Figure 5, it is clear that convergence is achieved. Thus we can obtain the posterior summary statistics.

**b. Numerical Summary**

In Table 2a and 2b, we have considered various quantities of interest and their numerical values based on MCMC sample of posterior characteristics for  $WEM(\beta, \lambda)$  model under Gamma priors. The Highest probability density (HPD) intervals are computed the algorithm described by Chen and Shao [22] under the assumption of unimodal marginal posterior distribution.

Characteristics	Chain 2	
	beta	lambda
Mean	0.2365198	0.02321709
Standard Deviation	0.009478293	0.006355558
Monte Carlo(MC) error	1.91E-4	1.318E-4
Minimum	0.1977	0.00783200
2.5th Percentile(P <sub>2.5</sub> )	0.2175	0.01281975
First Quartile (Q <sub>3</sub> )	0.2302	0.01865000
Median	0.2367	0.02245000
Third Quartile (Q <sub>3</sub> )	0.2430	0.02698000
97.5th Percentile(P <sub>97.5</sub> )	0.2544	0.03738000
Maximum	0.2718	0.05788000
Mode	0.2374240	0.02047058
95% Credible Interval	(0.2175, 0.2544)	(0.01282, 0.0373)
95% HPD Credible Interval	(0.2175, 0.2544)	(0.01191, 0.0356)

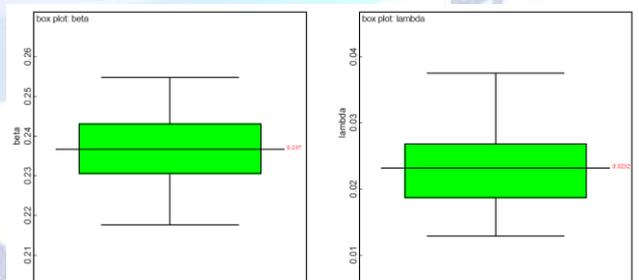
**Table 2a :** Num. summaries under Gamma priors for Chain 2.

Characteristics	Chain 1	
	beta	lambda
Mean	0.2368196	0.02310729
Standard Deviation	0.009519483	0.006369226
Monte Carlo(MC) error	1.531E-4	1.007E-4
Minimum	0.1998	0.00814300
2.5th Percentile(P <sub>2.5</sub> )	0.2178	0.01276975
First Quartile (Q <sub>3</sub> )	0.2307	0.01856000
Median	0.2369	0.02243000
Third Quartile (Q <sub>3</sub> )	0.2435	0.02677250
97.5th Percentile(P <sub>97.5</sub> )	0.2552	0.03747025
Maximum	0.2675	0.05703000
Mode	0.2368666	0.0198367
95% Credible Interval	(0.2178, 0.2552)	(0.01277, 0.0374)
95% HPD Credible Interval	(0.2180, 0.2552)	(0.01153, 0.0352)

**Table 2a :** Num. summaries under Gamma priors for Chain 1.

**c. Visual Summary**

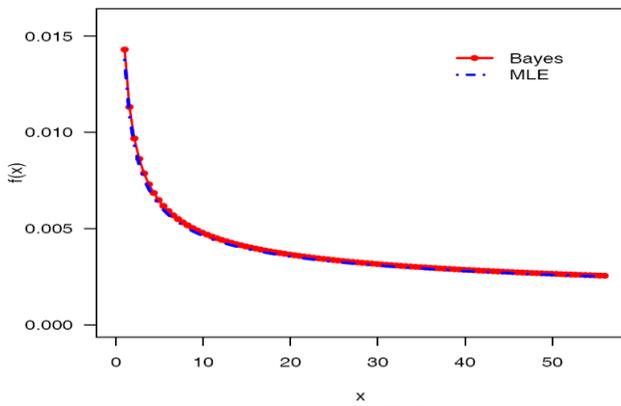
We use Box Plot for visual summary in Figure 6. The boxes represent inter-quartile ranges and the solid black line at the (approximate) centre of each box is the mean; the arms of each box extend to cover the central 95 per cent of the distribution - their ends correspond, therefore, to the 2.5% and 97.5% quantiles.



**Figure 6:** The boxplots for beta and lambda.

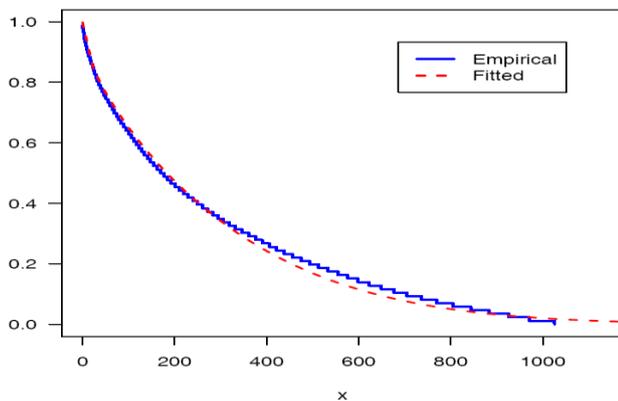
**Comparison with MLE**

For the comparison of Bayesian Estimates with MLE, we have plotted three graphs. In Figure 7, the density functions  $f(x; \hat{\beta}, \hat{\lambda})$  using MLEs and Bayesian estimates, computed via MCMC samples under gamma priors, are plotted.



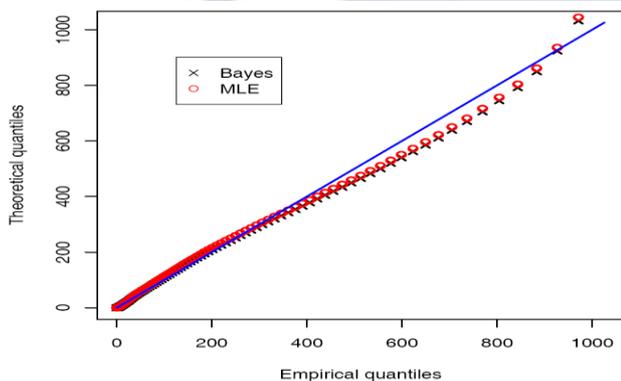
**Figure 7:** The density functions using ML and Bayesian estimates

Figure 8 exhibits the estimated reliability function (dashed line) using Bayes estimate under gamma priors and the empirical reliability function (solid line).



**Figure 8 :** The estimated reliability function (dashed line) using Bayes estimate under gamma priors and the empirical reliability function (solid line).

In Figure 9, Quantile-Quantile(Q-Q) plot of empirical quantiles and theoretical quantiles computed from MLE and Bayes estimates via MCMC samples under gamma priors, are plotted.



**Figure 9 :** Q-Q plot of empirical quantiles and theoretical quantiles computed from MLE and Bayes estimates.

It is clear from the Figures 7, 8 and 9, the MLEs and the Bayes estimates with respect to the gamma priors are quite close and fit the data very well.

## VI. CONCLUSION

This paper presented the two parameter Weibull Extension model and discussed the reliability analysis and some of its properties using developed functions in R software and also estimates of its parameters obtained on a complete sample by using Maximum Likelihood Estimator and Bayes Estimator using the Markov chain Monte Carlo (MCMC) method. We have applied the developed techniques on a real data set to show the performance of the estimation and to establish a comparison between these two proposed methodologies. Thus, the tools developed can be applied for full Bayesian as well as classical analysis of Weibull Extension model and show the MLEs and the Bayes estimates with respect to the gamma priors are quite close and fit the data very well.

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