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Involution on rings

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ABSTRACT

This paper deals with the concept of involution rings, involution ring properties and some important results and we discussed some important results in simple rings, minimal and maximal ideals with involution.

I. INTORDUCTION

Definition1.1. An involution on ring $R$is a unary operation $*$ on $(a^*)^* = a$, $(a+b)^* = a^* + b^*$ and $(ab)^* = b^*a^*$ for each $a, b \in R$. A ring with this opration is called a ring with involution or *-ring.

Definition1.2. Let R be a *-rings and let I be an ideal of a ring R. Then I is a *-ideal if I^* is sub set of I.(here we define for any non empty set S, we define $s^* = \{s^* : s \in S\}$)

Definition1.3.A non zero ideal I of an involution ring R which is closed under involution is termed as *-ideal.That is $I^* = \{a^* \in R | a \in I\} \subseteq I$

Definition1.4. Asub ring A of R is said to be a bi-ideal of R if $ARA \subseteq A$ and a *-bi ideal if in addition it is closed under involution of R.

Example let S be a ring and let X be an ideal of S with$X^2 = 0$. Let $R = \begin{pmatrix} S & X \\ X & 0 \end{pmatrix}$then with the standard matrix operation R is a ring and a matrix transposition is an involution, denoted by *. If S has a unity, then$R^2 = R$. If X is a minimal ideal of S,then $X = \begin{pmatrix} 0 & X \\ 0 & 0 \end{pmatrix}$ is a minimal ideal of R.

Definition1.5.$(S, +, .)$ is a semi- ring (a structure in which $(S, +)$ is a commutative semi group , $(S, .)$ is a semi group, and distributes over +) then $(S, +, .)$ is called an involution semi ring provided that *is an involution of S satisfying the identities $(x+y)^* = x^* + y^*$,$(xy)^* = y^*x^*$
Sometimes, semi ring may be equipped with a zero, and or an identities one. In such a case we require that the involution of satisfies $0^* = 0$ and $1^* = 1$.

Definition1.6.A right (left) ideals of I^* in a ring R^* is called *- minimal ideals if

i. $I^* \neq (0)^*$ and
ii. If J^* is a non zero (right/left) ideal of a ring R with involution contained in I^* then $I^* = J^*$.

Definition1.7. For a *-ring R the direct sum of all minimal *- ideal is denoted by $Soc^*(R)$, the *- scole of R. (if R has no *- minimal ideals then define $Soc^*(R)$ to be zero).and also we have $Soc^*(R)$ is the sub set of $Soc(R)$.

Definition1.8. Wesay R is *- simple R has no non zero proper ideals.

Definition1.9. An ideal A^* is said to be maximal ideals of the ring R with involution *, if

i. $A^* \neq R^*$
ii. For anyideals B^* is a sub set of A^*, either $B^* = A^*$ or $B^* = R^*$

Definition1.10. LetR be a ring with involution. For some $a \in R$, one may write i $I =< a >^* = Za + Za^* + aR + Ra + RaR + a^*R + Ra^* + Ra^*R$.

ii Clearly I is an ideal of R closed under involution and is called the principal *- ideal generated by a. One may deduce that $I =< a > +< a^* > = < a, a^* >$.

iii. A ring with involution * is a principal *- ideal ring if each *- ideal is a principal *- ideal. More over we say that a group G is strongly principal *- ideal ring group, if G is not nil ($G \neq 0$) and every ring R with involution satisfaying $R^2 \neq 0$, and $G = R^*$ is a principal *- ideal ring.

Definition1.11. Let R be *- ring. The intersection of all the nonzero ideals of R, called heart of R. and is denoted by H(R) so R is sub directly irreducible if and only if $H(R) \neq 0$. The intersection of all nonzero *- ideals of R is denoted by $H^*(R)$. Then R is *- sub directly irreducible if and only if $H^*(R) \neq 0$. If $H^*(R) = 0$, then R is a sub direct product of *- sub direct of irreducible ring.

Definition1.12.A *- ideal is said to be *- prime if when ever A, B are *- ideals such that $AB \subseteq P$ then$A \subseteq P, or\ B \subseteq P$. From this R is *- prime ring if and only if (0) is a *- prime ideal.

Corollary1.13. In everyinvolution ring $R, nR, R[n], R_t, R_p$ and the maximal divisible ideal D are *-ideals.

Proposition1.14. Let R be a *- ring. if I is *-minimal ideals of R ,then either

i. I is a minimal ideal of R ; or

ii. For any non zero ideal K of R, with K proper sub set of I,then $I = K \oplus K^*$,$K\&K^*$ are minimal ideals of R.further more $I^2 \neq 0$,then there are exactly two non zero ideals of R properly contained in I. If $R^2 = 0$, then only case(i)

Proof: Assume that I is not minimal ideal of R and let K be any non zero ideal of R with $K \subseteq I$ since $K + K^*$ and $K \cap K^*$ are *- ideals of R and contained in I .we have $K + K^* = I$ and $K \cap K^* = 0$. So $I = K \oplus K^*$.let Y be non zero ideal of R such that Y a sub set of K then by the above argument,$I = Y \oplus Y^*$ for any $k \in K$ we have $k = y_1 + y_2$ where $y_1 \in Y, y_2 \in Y^*$ and hence $k - y_1 = y_2$ thus $y_2 \in K \cap K^* = 0$,yielding $k \in Y$ and hence $K = Y$ so $K\&K^*$ are minimal ideals of R. Next let B be non zero ideal of R such that $B \subseteq I$ and Bneither K nor K^*.as with the argument for K, B is minimal ideal of R. So $BK = 0$. Similarly, BK^*, B^*K and B^*K^* are zero, yielding $I^2 = (B \oplus B^*)(K \oplus K^*) = 0$ when $R^2 = 0$ the set $\{k + k^* : k \in K\}$ is a *- ideal of R.

In part (ii) when $I^2 \neq 0$ we have $K^2 \neq 0$. Recall that a minimal ideal which is not square zero is a simple ring, so in the case we have K and K^* are simple rings. From $I = K \oplus K^*$ and $I \backslash K^*$ and $I \backslash K^* \approx K$ we see that K^*, and hence $K$are maximal ideals of the ring R . And in similar situation in the square,occur frequently enough to warrant looking more closely at the setting where the ring is the direct sum of two simple rings. Ideal of R and is contained I. So case(i) must hold when $R^2 = 0$.

Proposition1.15. Let$R$ be a *- ring and let I be a minimal ideal of R such that $I^2 \neq 0$. Then either (exclusively)

i. I is *- ideal of R; or

ii. $I \oplus I^*$ is a *- simple ring and hence is a *- minimal ideal of R.

Proof: Assumesthat $I \neq I^*$. Then minimally of I yields $I \cap I^* = 0$ and hence $I + I^* = I \oplus I^*$. Since$I^2 \neq 0$, the minimal ideals I and I^* are simple rings and the rings $I \oplus I^*$has only I and I^* as proper nonzero ideals. Let Y be a non zero proper *- ideal of the *-ring $I \oplus I^*$. So Y is I or I^* since Y is invariant under *, either case leads to the contradictory$I = I^*$. So $I \oplus I^*$ is a *- simple and consequently the only non zero *- ideal of R contained $I \oplus I^*$ is itself.

Example: Let Sbe a ring and let X be an ideal of S with $X^2 = 0$. If S has unity, then $R^2 = R$. If X is minimal ideal of S, then $X = \begin{pmatrix} 0 & X \\ 0 & 0 \end{pmatrix}$ is minimal ideal of R. observe that $X = 0$, $X \oplus X^*$is a *- minimal ideal of R.

Preposition1.16. Let R be a *- ring. Then$Soc(R) = T \oplus Soc^*(R) = T^* \oplus Soc^*(R)$, where T is the direct sum of minimal ideals K of R such that $K^2 = 0$ and $K \oplus K^*$ is not a *- minimal ideal of R.

Proof: Using a standard argument it can be shown that $Soc(R)$ is a direct sum of minimal ideals. Label this summands with ordinals so that $Soc(R) = \sum \oplus M_{\lambda}, \lambda \in A$ and $Soc^*(R)$ $\sum_{\alpha \trianglelefteq \lambda} \oplus M_{\lambda}, T = \sum_{\alpha \ntrianglelefteq \lambda} \oplus M_{\lambda},$. Then $Soc(R) = Soc^*(R) \oplus T$. If I is minimal ideal of Rand $I^2 \neq 0$ then I is the sub set of $Soc^*(R)$. Then $\alpha \leq \lambda$ we have $M_{\lambda}^{*2} = 0$ and $M_{\lambda} + M_{\lambda}^*$ is not a *-minimal ideals of R. since $T \cap Soc^*(R) = 0$ we have $T^* \cap Soc^*(R) = 0$ and consequently$Soc(R) = Soc^*(R) \oplus T^*$.

Proposition1.17. Let R be *- simple ring. Then either **R** is simple or R contains a maximal ideal K such that $R = K \oplus K^*$, K and K^* are simple rings $R^2 \neq 0$, and then the only proper non zero ideals of R are K and K^*.

Proof: Observe that if R is not simple, then R is *- minimal ideal of itself which is not a minimal ideal. So $R = K \oplus K^*$,where K and K^* are minimal ideals of R. if T is an ideal of the ring K then T is an ideal of , so minimality of K yields $T = 0$ or $T = K$. Thus K is a simple ring. Since $R/R^* \approx K$, we have K^***is a** maximal ideal of R. similarly we obtain $K^*$is a simple ring and K is a maximal ideal of R. since $R^2 = 0$ implies the existence of non zero proper *-ideal of R, we must have $R^2 \neq 0$ then K and K^* the only non zero proper ideals of R.

Theorem1.18. Let R be *- ring and let M be a *-maximal ideal of R. if M is not maximal ideal of R, then there exist a maximal ideal K of R such that;

i. $K + K^* = R$ and $K \cap K^* = M$; so $R\backslash K^*$ is a simple and $R\backslash R^* \approx K\backslash M$
ii. $R^2 + M = R$ and the only proper ideal of R which properly contains M are K and K^*;
iii. K and K^* are *- essential in R;
iv. Either M is essential as an ideal in the ring R, or ther exist a maximal ideal of R such that $K^* = M \oplus I$ and $R = I \oplus I^* \oplus M$.

Proof: Since $R = R\backslash M$ is a *- simple ring which is not simple, we have that there exists a proper ideal K of R with M proper sub set of K such that $K\backslash M \oplus (K\backslash M)^*$, with $K\backslash M$ and $(K\backslash M)^* = K^*\backslash M$ being both maximal as ideals are and simple as rings. Thus K and K^* are maximal ideals of R, $K + K^* = R$, $K \cap K^* = M$ since $R^2 = R$, the only proper non zero ideals of R are $K\backslash M$ and $K^*\backslash M$. Then there are no proper ideals of R that properly contain M except K and K^*.For a purpose of contradiction assume there exist a non zero *-ideal T of R such that $T \cap K = 0$ also $T \cap K^* = 0$. Then $TK = 0 = TK^*$ and hence $TR = T(K + K^*) = TK + TK^* = 0$. Since $T \cap K = 0$ also we have $R = M + T = M \oplus T$. Hence $R\backslash M \approx T$, and T is a *- simple ring which not a simple ring.also we have$T^2 \neq 0$, a contradiction to $TR = 0$. Thus K^* is *- essential in R, which immediately yields that K^* also.To establish (iv) consider M to be not essential in the set of ideals of the ring R. observe that if K is essential in the set of ideals of the ring R, then so K^*, and consequently the non zero intersection $M = K \cap K^*$ would also be essential. Thus there exists a non zero ideal I of R such that $I \cap K = 0$. The only ideals of R which contains M are M,K,K^* and R. so either $I \oplus M = K^*$ or $I \oplus M = R$. In the former case, from $I \approx M\backslash K^*$ we have that I is a simple ring and hence I is a minimal ideal of R. finally consider $I \oplus M = R$ for each $k \in K$ there exists $\in I$,$m \in M$ such that $k = i + m$. But $i = k - m$ is in K, forcing $i = 0$. So $k = m$. Since k is arbitrarily this yields the contradictory $M = K$.Thus $I \oplus M \neq R$.

Theorem1.19. Let R be *- ring. If M is a maximal ideal of the ring R, then either (exclusively);

i. $M \cap M^*$ is a maximal ideal of R; or
ii. For any *- ideal K of R such that $M \cap M^* \subseteq K \subseteq R$, then K is a maximal ideal of R and there is no ideal of R properly contained between $M \cap M^*$ and K, and between $M \cap M^*$ and M.

Proof:(i). Assume that $M \cap M^*$ is not maximal ideal of R. then $M \neq M^*$. let K be any *-ideal of R such that $M \cap M^* \subseteq K \subseteq R$. From $R\backslash M^* = (M + M^*)\backslash M^* \approx M(M \cap M^*)$, we have that $M(M \cap M^*)$ is a simple ring. Thus $M \cap M^*$ is a maximal ideal of the ring M and there are no ideals of R properly contained between $M \cap M^*$ and M. from $M \cap M^*$ sub set of $M \cap K$ also sub set of M, we have either $M \cap K = M$ or $M \cap K = M \cap M^*$. The former yields M sub set of K and hence $M = K$, a contradiction to M is *- ideal. Consider $M \cap K = M \cap M^*$.then K is not sub set of M and $R\backslash K = (M + K)\backslash K \approx M(M \cap M^*)$. Thus $R\backslash K$ is a simple ring and consequently K is maximal ideal of R. from $R\backslash M = (K + M)\backslash M \approx K\backslash(K \cap M) = K(M \cap M^*)$, we see that $K\backslash(M \cap M^*)$ is a simple ring and hence there are no ideals of R properly contained in between $M \cap M^*$ and K.

Lemma1.20. Let $G = H \oplus K$, $H \neq 0, K \neq 0$ be a strongly principal *- ideal ring group. Then H and K are either both *- cyclic or both nil.

Proof: Suppose that H is not nil. Let S be a *- ring with $S^+ = H$ and $S^2 \neq 0$ and let T be the zero ring on K. the ring direct sum $R = S \oplus T$ is the ring with involution satisfaying $R^+ = G$ and $R^2 \neq 0$. Since T is a *- ideal in R, $T = < x >^*$. Clearly $K = T^+ = (x)^*$. Therefore K is not nil interchanging the roles of H and K yields that His *- cyclic.

Corollary1.20. Let $G = H \oplus K, H \neq 0, K \neq 0$ be strongly principal *- ideal ring group. Then H and K are *- cyclic.

Proof: It suffices to negate that H and K are both nil. Let $R = (G .)$ be a ring with involution satisfying$R^2 \neq 0$.

Case (i): suppose that $R^2 \subseteq K$. There exist $h_0 \in H, k_0 \in K$, such that $R = < h_0, k_0 >^*$. Let $h \in H$, since $h \in R$ there exist integers n and m, and $x \in R^2$ such that $h = n(h_0 + k_0) + m(h_0 + k_0)^* + x$However , $x \in K$, so $h = nh_0 + mh_0^*$ and H is *-cyclic, contradicting the fact that H is nil.

Case (ii): suppose that $R^2 \neq K$. For all $g_1, g_2 \in G$ define $g_1 o g_2 = \pi_H(g_1 o g_2)$, where π_H is natural projection of G on to H. since

$$(g_1 o g_2)^* = (\pi_H(g_1 o g_2))^* = \pi_H(g_1 o g_2)^* = \pi_H(g_2^* o g_1^*) = g_2^* o g_1^*$$

Hence $S = (G, o)$ is a ring with involution satisfying $S^2 \subseteq H$. The argument employed in (i) yield that K is *- cyclic with contradicts the fact that K is nill.

Theorem1.21. There are no mixed strongly principal *- ideal ring groups.

Proof: Let G be a mixed strongly principal *- ideal ring group. G is decomposable, so by lamma (that is let $G = H \oplus K, H \neq 0, K \neq 0$ be strongly principal *- ideal ring group. Then H and K are either both *- cyclic or both nill. So $G = H \oplus K, H \neq 0, K \neq 0$ with H and K both *- cyclic or both nill.

1) Suppose that H and K are both nill. There are no mixed nil groups by so, we may assume that H is a torsion group, and that K is torision free. Let R be *- ring with $R^+ = G$ and $R^2 \neq 0$. Clearly H is a *- ideal in R and so $H =< h >^*$.

Let $|h| = m$, then $mH = 0$. H is divisible, there fore not bounded, a contradiction.

2) Suppose that $H = (x)^*$ and $K = (e)^*$ with $|x| = n$, and $|e| = \infty$. The products $x^2 = xe = ex = e^* = x = ex^* = xe^* = x^*e = 0$ and $e^2 = ne$ induce a *- ring structure R on G satisfying $R^2 \neq 0$. Therefore there exist integers s and t such that $R =< sx + te >^*$. Every $y \in R$ is of the form $y = m_y sx + (m_y + u_y n) + m_y sx^* + (m_y^* + u_y^* n)te^*$, with m_y, m_y^*, u_y, and u_y^* integers. In particular $(m_e + u_e n)t = 1$. Hence $t = \pm 1$. Therefore, $m_x + u_x n = 0$ and so $n \backslash m_x$, however , $x = m_x sx = 0$, is a contradiction.

Theorem1.22. Let G be a mixed group. Then

1. If G is a principal *- ideal ring group, then G_t is bounded and $G \backslash G_t$ is a principal *- ideal ring group.
2. Conversely, if G_t is bounded and if there exists a unital principal *- ideal ring with additive group $G \backslash G_t$, then G is a principal *- ideal ring group.

Proof: 1) Let R be a principal *- ideal ring with $R^+ = G$. Since G_t a *- ideal in R, $G_t =< x >^*$ and $nG_t = 0, n = |x|$ now $G = G_t \oplus H$ and $H \cong G \backslash G_t$. Now $R =< a + y >, a \in G_t, 0 \neq y \in H$.

Suppose that $R^2 \subseteq G_t$ and let $h \in H$. Then there exist integers k_n, k_n^* such that ,
$h = k_n y + k_n^* y^* + b$ with $b \in R^2$ since $R^2 \subseteq G_t$, $b = 0$ and $h = k_n y + k_n^* y^*$. Therefore $H = (y)^*$. H is a principal *- ideals ring group.

If $R^2 \nsubseteq G_t$, then $R = R \backslash G_t$ is a principal *- ideal ring with $R^+ \cong G \backslash G_t$, and $R^2 \neq 0$.

2) Conversely, suppose that G_t is bounded, and that there exists a unitals principals *- ideals ring S with unity and * is the identity involution such that $S^+ = G_t$. Let $R = SI = (I \cap S) \oplus (I \cap T)$. T with e, f the unities of S and T, respectively. Then R is a ring with involution *.

Let I be a *- ideal in R, then $I = (I \cap S) \oplus (I \cap T)$
Now $I \cap S$***Δ*****S** **and** so $I \cap S =< x >^*$. Similarly $I \cap T =< y >^*$.

Clearly $< x + y >^* \subseteq I$ however, $x = e(x + y) \in < x + y >^*, x^* = e(x + y)^* \in < x + y >^*$, and $y = f(x + y) \in < x + y >^*, y^* = f(x + y) \in < x + y >^*$

Hence we conclude that $I =< x + y >^*$.

Theorem1.23. Let G be a torision free strongly *- ideal ring group. Then G is either indecomposable, or is the direct sum of two nil groups.

Proof: It suffices to negate that $G = (x_1)^* \oplus (x_2)^*, x_i \neq 0, i = 1,2$

Suppose that it is so the product : $x_i x_j = 3x_i$ and $x_i^* x_j = 0$ for $i = j = 1,2$, $x_i x_j = x_i^* x_j = 0$, for $i \neq j$ induce a ring structure R of G with involution *- satisfaying $R^2 \neq 0$. Therefore there exist non zero integer k_1, k_2 such that $R = < k_1 x_1 + k_2 x_2 >^*$.Every $x \in R$ is of the form:

$\boldsymbol{x = (rx + 3sx)k_1 x_1 + (rx + 3tx)k_2 x_2 + (r_x' +}$ ***3sx'k1x1*+rx'+3tx'k2x2*** **where** ***rx***,*rx',sx,tx, sx',tx'* are integers. From $rx_1 + 3sx_1 = \pm 1$, it follows that $rx_1 \equiv \pm 1 (mod 3)$. However $rx_1 + 3tx_1 = 0$ implies $rx_1 \equiv 0 (mod 3)$ which is a contradiction.

Theorem1.24. Let G be torision strongly principal *- ideal ring group. Then G is a *- cyclic group or $G = (x_1)^* \oplus (x_2)^*$, with $|x_i| = p$, a prime , where $i = 1,2$.

Proof: Suppose that G is a strongly principal *- ideal ring group. Let G be indecomposable .then $G \cong Z_{pn}$, where p is a prime , $1 \leq n \leq \infty$. If $n = \infty$, then G is divisible and so G is nil, which is a contradiction. Hence G is cyclic and so *- cyclic.

Next suppose that $G = H \oplus K, H \neq 0, K \neq 0$, either H or K are both *- cyclic or both nil. If H and K are nil, then they are both divisible, so G is nil which is again a contradiction. Therefore $G = (x_1)^* \oplus (x_2)^*$ with $|x_i| = n_i i = 1,2$. If $(n_1, n_2) = 1$, then G is *- cyclic . other wise, let p be a prime divisor of (n_1, n_2). Then $G = (y_1)^* \oplus (y_2)^* \oplus H$ with $|y_i| = pm_i, i = 1,2$ and $1 \leq m_1 \leq m_2$. Since $(y_1)^* \oplus (y_2)^*$ is neither *- cyclic nor nil, $H = 0$. The product $y_i y_j = p_2^m - ly_2, y_i y_j^* = 0$ where $i, j = 1,2$ induce a *- ring structure R on G with $R^2 \neq 0$. therefore, $R =< s_1 y_1 + s_2 y_2 >^*$, where s_1 *and* s_2 are integers. Every element $x \in R$ has the form :

$x = k_x s_1 y_2 + (k_x s_2 + m_x p_{2-1}^m) y_2 + k_x s_1 y_1' + (k_x' s_2 + m_x p_{2-1}^m) y_2^*$.

Preposition1.25. Let R be *- ring. Then the following are equivalent;

i. $(H^*(R))^2 \neq 0$

ii. R is *- prime ring with at least one minimal ideal ring.

Proof: Let $H^* = H^*(R)$. Assume that $(H^*)^2 \neq 0$. Then H^* is minimal *- ideal of R. either H^* minimal of the ring R, or $H^* = K \oplus K^*$ where K and K^* are minimal ideals of R. in either case, if A and B are non zero *- ideals of R, then $H^* \subseteq A$, $H^* \subseteq B$ and hence $(H^*)^2 \neq 0 \subseteq AB$. So R is *-prime ring.
Conversely assume that (ii). Let I be a fixed minimal *-ideal of R and let A be any non zero *-ideal of R. then $AI \neq 0$. So $A \cap I = I$. Thus $I \subseteq A$ for each *-ideal A of R, yielding $H^* = I$. Since *- prime we have $(H^*)^2 \neq 0$. From this we conclude that every prime ideal is maximal ideal.

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