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Involution on groups

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ABSTRACT

This paper deals with the concept of involution, involution properties and some important results and we discussed about a group with involution. Especially, important results in cyclic group with involution.

I. INTORDUCTION

The paper reveals an involution on a certain classes of groups. Aburewash,U.A(2000) made an investigation on involution ' rings and studied its properties .Feigalstock,S (1983) studied about additive groups of rings . Further Feigal stock and schlussel (1978) made an attempt on the properties of principal ideals and Nietheriangroup.FuchesL(1973,1970) made an account an infinite abelian groups. Further the same notion was studied by Hungerford (1974).Lio(1990) contributed on the structure of semi prime prime involution rings.Infat Robinson (1991) provided a detailed information about the theory of groups.

Definition1.1. A function $f: A \to A$ (that is a unary operation on the set) is called an involution if $f(f(x)) = x$ holds for all x in A.

Definition1.2.An element x of R is called symmetric if and only if $x^* = x$ and skew if $x^* = -x$

Definition 1.3. Let a group G is decomposed in to its sub groups as$G = H \oplus K$. If G has an involution *then * is said to be changeless involution in case $g* = (h^*, k^*)$, $\forall g = (h,k)\epsilon H \oplus K$

Definition1.4. A torsion free group G has rank r if r is the number of elements in a maximal independent set in G.that is there exists an independent set with $r$elements and any larger set is dependent. [The rank of $G$is also equal to $\dim_Q(G \oplus Q)$.]

Definition1.5. let $f: A \to B$ be a group or a ring homomorphism. If A and B are equipped with some involution $*A$ and $*B$ such that $f(a^{*A}) = [f(a)]^{*B}$ then we say that f is an involution preserved homomorphism. If f is an involution preserved isomorphism then we will write

$A \cong^* B$.it is clear that *- sub groups and *-ideals are preserved under such isomorphism. Moreover If$A \cong B$ as a group or a ring, then every involution on A induced an involution on$B$.

Definition1.6. An involution * is said to be *-semi groups or (involution semi groups) if which satisfies $(xy)^* = y^*x^*$and$(x^*)^* = x$.

Definition1.7. if A represents a linear involution, then $x \to A(x-b)+b$ is an affine involution.

Definition 1.8. in the case that the eigen space for eigen value one is the orthogonal complement of that for eigen value -1.that is every eigen vector with eigen value 1 is orthogonal to every eigen vector with eigen value -1, such an affine involution is an isometric.

Definition1.9.A group G together with a urinary operation * is said to be a group with involution, in case,for all $a,b \in G$,$(a^*)^* = a$ and $(a+b)^* = a^* + b^*$

Definition1.10.An involution semi group is triple $s(.,*)$ such that $(S,.)$ is a semi group while *-is an involution on S such that $(xy)^* = y^*x^*$ holds for all $x,y \in S$ and $(x^*)^* = x$

Definition1.11.A group G is said to be nil, in case the only ring R with $G = R^+$ is the zero ring.

Definition1.12.A non zero sub group H of an involution group G which is closed under involution is termed as a *-sub group and denoted by $(H \leq G)$.

Corollary1.13. Let $R = A \oplus B$ where A and B are rings in which B is commutative then R has a (changeless) involution if and only if G has an involution.

Proofs: One way is clear assume that A has an involution A^*. Define $*R$on $R$by $r^{*R} = (a^{*A}, b)$ then $*R$is a uniray opration on R and for $r_1 = (a_1, b_1)$, $r_2 = (a_2, b_2)$ where $r_1, r_2 \epsilon R$,

$$(r_1 r_2)^{*R} = ((a_1 a_2)^{*A}, b_1 b_2) = (a_2^{*A} a_1^{*A}, b_2 b_1) = r_2^{*R} r_1^{*R}$$

Theorem1.14. Let $(R,*)$ be a ringwith involution and let F be a generalized derivation of R such that $F(x0y) = x0y$ for all x, y in a non zero *-ideal $I$of R. If R is $*$ - prime, then$R$ is commutative.

Proof: Assume that F is a generalized derivation associated with a derivation d.If F is trival then the condition $F(x0y) = x0y$ for all $x, y \in F$ reduces to $x0y = x0y$for all $x, y \in F$.in which case our theorem. Now when the associated derivation $d = 0$, then F is nontriaval left multiplier and using the collorarly (let $(R,*)$ be a ring with involution and let F be a non trivial left multiplier such that $F(x,y) = (x,y)$for all x, y in a non zero *-ideal I of R .if R is *- prime, then R is commutative.)we get the required result.If $d \neq 0$ indeed,neither d computes with *-nor $2-$ torsion freeness is necessary.it is proved that if a prime ring R ring admits a non zero left multiplier H, with $H(x) \neq x$ for all x in a non zero ideal I of R, such that $H(x0y) = x0y$ for all $x, y \in I$, then R commutative.however it's$H(x) \neq x$ for all $x \in I$ can be replaced by there exists $x \in I$ such that $H(x) = x$ that is H is nontrivial.

Lemma1.15. Let G be a group with an involution *.then the following sub groups of G are closed under the involution *

a). $nG, \forall n \in Z$

b). the torsion subgroup G_t of G

c). for any prime P,every p- primary subgroup G_p of G.

d). the maximal divisible subgroup of G.

e). the subgroup $G[m] = \{g \in G / mg = 0\}$ of G ,for some integer m.

Lemma 1.16. a). Every direct sum of involution groups are an involution group.

b). every direct summand of a group with a changeless involution is an involution group.

c). If a direct summand of a group has an involution then the group has an involution.

Proof: a) Let $G = H \oplus K$ where H and K are groups with involution $*H$ and $*K$ respectively,then for every $g = (h, k) \in G$ where $h \in H$ and $k \in K$, define the involution $*G$ on G by $g^{*G} = (h^{*H}, k^{*K})$. Because of the unique representation of each element ,$*G$ becames a unary operation on G .further , $(g^{*G})^{*G} = ((g^{*H})^{*H}, (g^{*K})^{*K}) = (h, k) = g$ .Assume that $g_i \in G$ with $g_i = (h_i, k_i)$where $h_i \in H$,and $k_i \in K$,then

$$(g_1 + g_2)^{*G} = ((h_1 + h_2)^{*H}, (k_1 + k_2)^{*K}) = ((h_1^{*H} + h_2^{*H}), (k_1^{*K} + k_2^{*K})) = (h_1^{*H}, k^{*K}) + (h_2^{*H}, k_2^{*K}) = g^{*G_1} + g^{*G_2}$$

Hence $*G$is an involution on G, it is in fact the changeless involution on G .the proof can analogously be extended to finite as well as to arbitrarily direct sums.

b. Let $G = H \oplus K$ set $H' = H \oplus 0$ and $K' = K \oplus 0$ clearly $H'$and K' are dire summand and sub groups of G. Assume that $*$ is the changeless involution on G. then $*\backslash H'($ involution on G restricted to $H)$ is an involution on $H'$and $*\backslash K'$ is an involution onK'.also $H^* \cong H'$ and $K \cong^* K'$
Hence (b) is proved

c. Let $G = H \oplus K$ and H be a group with an involution $*H$ then for every $g = (h, g) \in G$
Where $h \in H$ and $k \in K$,define an opretion $*G$ on G by $g^{*G} = (h^{*H}, k)$. Clearly,$*G$ is the changeless involution on G.

Preposition1.17. Let $x_1$and x_2 be elements of a group G such that a prime $p \backslash (|x_1|, |x_2|)$. If $G = (x_1)^* \oplus (x_2)^*$, then there exist $y_1, y_2 \in G$ such that $(y_1)^* \leq (x_1)^*$ and $(y_2)^* \leq (x_2)^*$

Proof: Let $G = (x_1) + (x_2)^* \oplus (x_2) + (x_2)^*$ .If p is a prime such that $p \backslash |x_1|$ then there exists $y_1 \in (x_1)$ such that $p \backslash |y_1|$ and $|y_1|$ divides $|x_1|$. Consequently, $y_1 \leq (x_1)$ and $(y_1^*) \leq (x_1^*)$ similarly there is $y_2 \in x_2$ such that $(y_2) \leq (x_2)$and$(y_2^*) \leq (x_2^*)$. Hence it is conclude that $(y_1) + (y_1^*) \leq (x_1) + (x_1^*)$and $(y_2) + (y_2^*) \leq (x_2) + (x_2^*)$ that is $(y_1)^* \leq (x_1)^*$ and
$(y_2)^* \leq (x_2)^*$.

Proposition 1.18. *Let G = H ⊕K. If H and K are *-cyclic groups suchthat* $H = (a)^*, K = (b)^*$, $(|a|, |b|) = 1$. *Then G is *-cyclic.*

Proof: The given condition $(|a|, |b|) = 1$ implies that $(a) \oplus (b)$ is a cyclic group generated by (a, b) and $(a^*) \oplus (b^*)$ is a cyclic group generated by (a^*, b^*). But,

$$G = H \oplus K = (a) + (a^*) \oplus (b) + (b^*) = (a) \oplus (b) + (a^*) \oplus (b^*).$$

Hence G is *-cyclic with $G = ((a, b))^*$.

Proposition 1.19. *Let G be an additive abelian group,* $G = H \oplus K$, *and let H and K are cyclic subgroups of G. If* $(|H|, |K|) \neq 1$, *then*
(*a*) *G has exactly four involutions, namely:*

$$g* = (h, k), g* = (-h, k), g* = (h, -k), g* = (-h, -k) \text{ and } g* = (h, k).$$

(*b*) Every subgroup of G is closed under involution

Proof: (*a*) Hand Kare *-subgroups. Since Hand KareCyclic, Hand K,each, has two involutions; the identity involution and *:$a \to -a$. Hence again by Lemma 2.3, G has exactly the given four involutions.

(*b*) any subgroup H of G is a direct sum of twocyclic subgroups, or it is cyclic. Hence by (*a*), $H$is a *-subgroup.

Proposition1.20. Let R be a ring with involution such that $R^+ = G$. Then R has only the identityinvolution in case any one of the following holds:

(a). G is a cyclic group.

(b). G is a direct sum of cyclic subgroups.

Proof: (a) Let Gbe cyclic. Since $R$is an involution ring, G has either theidentity involution or the involution$* : a \to$

$-a$ .However, $-(ab) \neq (-b)(-a)$,for all $a, b \in R$. Hence R has the identity involution only.

(b) If $G = H \oplus K$, and H and K are cyclic subgroups of G, then byProposition 3.1, G has four involutions. But then again by (1), G has onlyone involution.

Preposition1.21. Let $G = H \oplus K$ .if H and K are *-cyclic groups such that $H = (a)^*, K = (b)^*, (|a|, |b|) = 1$then G is *-cyclic.

Proof:The given condition $(|a|, |b|) = 1$ implies that $(a) \oplus (b)$ is a cyclic group generated by (a, b) and $(a^*) \oplus (b^*)$ is a cyclic group generated by (a^*, b^*) ,but

$$G = H \oplus K = (a) + (a^*) \oplus (b) + (b^*)$$
$$(a) \oplus (b) + (a^*) \oplus (b^*)$$

Hence G is *-cyclic with $G = ((a, b))^*$

Preposition1.22. If G is a *-cyclic group, then any *-subgroup of G is a *-cyclic sub group.

Proof: A *-cyclic group is either torision or torision free. First assume that G is torision free and let $G = (a)^* = (a, a^*)$. If $(a) \cap (a^*) \neq 0$. Then $na^* = ma \neq 0$ for some integers m and n. This implies $na = ma^*$ so $na - na^* = ma^* - ma$, from which
$n(a - a^*) = m(a^* - a) = -m(a - a^*)$ and also $(n + m)(a - a*=0$.since G is torision free $a-a*=0$ implies that $a = a^*$.hence $(a) \cap (a^*) = 0$ and $G = (a) \oplus (a^*)$.

Secondly assume that G is torision, $G = (a) + (a^*)$ and $|a| = |a^*| = k$. Let $g \in G$,
$g = ma + na^*$ for some integer m, n since $k(ma + na^*) = 0$ it follows that $|g| \leq k$ and a, a^* have maximal orders. Hence $G = (a) \oplus (a^*)$

Thus in both cases $G = (a) \oplus (a^*)$ .if $H \leq^* G$ then $H = (b) \oplus (c)$, where $(b) \leq (a)$ and $(c) \leq (a^*)$. Then $(b) = ma$ and $(c) = (na)^*$. hence $H = (ma) \oplus (na)^*$.since H is *-sub group, $ma^* + na \in H$.but $ma^* \in (na)^*$, so $m > n$ and $na \in (ma), so\ n > m$. Therefore $n = m$ and $H = (na) \oplus (na^*)$.

Hence $H = (na)^*$ and H is a *-cyclic subgroup.

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